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Computation of Unsteady Turbulent Boundary Layers with Flow Reversal and Evaluation of Two Separate Turbulence Models

Tuncer Cebeci and Lawrence W. Carr

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COMPUTATION OF UNSTEADY TURBULENT BOUNDARY LAYERS WITH FLOW REVERSAL AND EVALUATION OF TWO SEPARATE TURBULENCE MODELS

bу

Tuncer Cebeci* and Lawrence W. Carr

SUMMARY

Recently a new procedure, which solves the governing boundary-layer equations with Keller's box method, has been developed for calculating unsteady laminar flows with flow reversal [1]. In regions where the streamwise velocity contains flow reversal, the solution scheme was modified by a procedure which accounted for the downstream influence. With this modification, the unsteady flow over a circular cylinder started impulsively from rest was successfully calculated to values of time and space greater than in any previous solutions. An examination of unsteady separation for laminar flow was made and revealed that the unsteady boundary layer for that flow, even at large times, was free of singularities.

In this report we extend the method of ref. [1] to turbulent boundary layers with flow reversal. Using the algebraic eddy viscosity formulation of Cebeci and Smith [2], we consider several test cases to investigate the proposition that unsteady turbulent boundary layers also remain free of singularities.

Since the solution of turbulent boundary layers requires a closure assumption for the Reynolds shear-stress term and the accuracy of the solutions depend on this assumption, we also perform turbulent flow calculations by using the turbulence model of Bradshaw, Ferriss and Atwell [3]; we solve the governing equations for both models by using the same numerical scheme and compare the predictions with each other, restricting the comparisons to cases in which wall shear is positive.

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The study reveals that, as in laminar flows, the unsteady turbulent boundary layers are free from singularities but there is a clear indication of rapid thickening of the boundary layer with increasing flow reversal. The study also reveals that the predictions of both turbulence models are the same for all practical purposes.

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I. INTRODUCTION

The prediction of unsteady turbulent boundary layers with flow reversal is of importance in a number of aerodynamic problems, notably in dynamic stall, buffeting and gust studies. However, some of the more popular turbulence models implicitly assume that the wall shear is positive and their extension to unsteady flows with flow reversal is not easy. It requires modifications to the functional form of the law of the wall and to the manner in which the wall shear is determined. Two near-wall assumptions are considered here. In the first, the near wall grid point is located in the logarithmic region and the law of the wall is used to link the flow properties at this grid point to the wall. In the second, a Van Driest formulation due to Cebeci and Smith [2] is used; this implies that the grid point closest to the wall will occur in the viscous sublayer.

A further aspect of these flows of current interest is the possibility of a singularity occurring in the reversed-flow region. Examples of this phenomenon have also been reported in laminar flows but, in earlier studies Cebeci [1,4] and Bradshaw [5] have shown that the occurrence is not a feature of the governing equations but is due to the limitations of the numerical procedure used. We shall demonstrate that, for the examples we study, there is no indication of such a singularity in turbulent flow either but there is a clear indication of rapid thickening of the boundary layer.

In addition to the examination of wall functions, we have also considered two turbulence models for unsteady flows without flow reversal. The algebraic eddy-viscosity formulation of Cebeci and Smith (CS) is compared with the transport model of Bradshaw, Ferriss and Atwell [3] (BF). Calculations were performed to determine whether the representation of unsteady flows with strong pressure gradients requires that account be taken of transport of turbulence quantities. As will be shown, the predictions with both models are nearly identical for both steady and unsteady flows with and without strong pressure gradient.

The report has been prepared with six main sections describing, respectively, the governing equations, the numerical procedure, the results, concluding remarks, references and the computer program which uses only the CS model.

II. GOVERNING EQUATIONS

The continuity and momentum equations can be written for two-dimensional unsteady incompressible laminar or turbulent thin shear layers as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t} + u_e \frac{\partial u}{\partial x} + \frac{\partial \tau}{\partial y}$$
 (2)

Here

$$\tau = v \frac{\partial u}{\partial v} - \overline{u^* v^*}$$
 (3)

and we recall that u' and v' denote fluctuations about the ensemble-average velocity; u' and v' are zero in unsteady laminar flow, and v $\partial u/\partial y$ is negligible outside the viscous sublayer in a turbulent flow. These equations are subject to the usual boundary conditions, which in the case of boundary layers are

$$y = 0,$$
 $u = v = 0;$ $y + \delta u + u_{\rho}(x,t)$ (4)

The presence of the Reynolds stress term, -u'v' introduces an additional unknown to the system given by Eqs. (2) to (4). In this report we present calculations using two different turbulence models. One is an algebraic eddy-viscosity formulation developed (for steady flows) by Cebeci and Smith and the other is a transport-equation model developed by Bradshaw, Ferriss and Atwell. In the CS model, we write Eq. (3) as

$$\tau = (v + \epsilon_{\rm m}) \frac{\partial u}{\partial y} \tag{5}$$

with two separate formulas for ϵ_m . In the so-called inner region of the boundary layer $(\epsilon_m)_i$ is defined by the following formula:

$$(\varepsilon_{\rm m})_{\rm j} = \{0.4y[1 - \exp(-y/A)]\}^2 \left| \frac{\partial u}{\partial y} \right|$$
 (6)

where

$$A = 26vu_{\tau}^{-1}[1 - 11.8(p_{t}^{+} + p_{x}^{+})]^{-\frac{1}{2}}$$
 (7a)

$$u_{\tau} = \left(\frac{\tau_{w}}{\rho}\right)^{\frac{1}{2}}, \quad p_{t}^{+} = \frac{v}{u_{\tau}^{3}} \frac{\partial u_{e}}{\partial t}, \quad p_{x}^{+} = \frac{vu_{e}}{u_{\tau}^{3}} \frac{\partial u_{e}}{\partial x}$$
 (7b)

In the outer region $\;\varepsilon_{m}^{}\;$ is defined by the following formula

$$(\epsilon_{\rm m})_{\rm o} = 0.0168 \int_{0}^{\infty} (u_{\rm e} - u) dy$$
 (8)

The boundary between the inner and outer regions is established by the continuity of the eddy-viscosity formulas.

In the BF model, which is used <u>only</u> outside the viscous sublayer, we assume $\tau = -\overline{u^1v^1}$ and write a single first-order partial-differential equation for it; the equation was originally developed from the turbulent energy equation but can be equally well regarded as an empirical closure of the exact shear-stress transport equation. **This re**ads

$$\frac{D\tau}{Dt} = \frac{\partial \tau}{\partial t} + u \frac{\partial \tau}{\partial x} + v \frac{\partial \tau}{\partial y} = 2a_1 \tau \frac{\partial u}{\partial y} - \frac{\partial}{\partial y} (\tau V_{\tau}) - 2a_1 \frac{\tau^{3/2}}{L}$$
 (9)

Here a_1 is a dimensionless quantity, V_{τ} is a velocity and L is the dissipation length parameter, specified algebraically by $L/\delta = f(n)$ with $n = y/\delta$ and f(n) given from an analytic fit to an empirical curve by

$$f(n) = \begin{cases} 0.4n & \eta < 0.18 \\ 0.095 - 0.055(2n - 1)^2 & 0.18 \le \eta < 1.1 \\ 0.016 \exp[-10(n - 1.1)] & \eta \ge 1.1 \end{cases}$$
 (10)

In a more advanced version of this turbulence model [6] L itself is determined from a transport equation.

The turbulent transport velocity V_{τ} , nominally $(p'u' + u'v'^2)/u'v'$ is proportional to a velocity scale of the large eddies and is chosen to be

$$V_{\tau} = 2a_{1} \frac{\tau_{\text{max}}}{u_{e}} g(\eta)$$
 (11)

where g(n) is given by

$$g(\mathbf{n}) = \begin{cases} 33.3n^{2}(0.184 + 0.832n) & \eta < 0.5 \\ 33.3n^{3}(0.368 + 2.496n^{3}) & 0.5 \le n < 1.0 \\ 18.7n + 14.60 & \eta \ge 1.0 \end{cases}$$
 (12)

In the BF model equations, the inner boundary conditions for (1), (2) and (9) are applied outside the viscous sublayer, usually at $y_1 = 50v/u_{\tau}$. In the steady-flow study reported in [7], these boundary conditions are:

$$u_1 = v_{\tau} \left(\frac{1}{\kappa} \ln \frac{y_1 u_{\tau}}{v} + 5.2 \right)$$
 (13)

$$v_{1} = -\frac{u_{1}y_{1}}{u_{\tau}} \frac{\partial u_{\tau}}{\partial x}$$
 (14)

$$\tau_{\parallel} = \tau_{W} + \frac{1}{\rho} \frac{\partial p}{\partial x} y_{\parallel} + \alpha \star \frac{\partial \tau_{W}}{\partial x} y_{\parallel}$$
 (15)

Here v_1 is evaluated from the continuity equation (1), and α^* is evaluated from (1) and (2) on the assumption that the velocity u is given by

$$\frac{u}{u_{\tau}} = \phi\left(\frac{u_{\tau}y}{v}\right) \tag{16}$$

for $0 < y < y_1$; Eq. (13) is, of course, a special case of (16). The evaluation of α^* is discussed in Ref. [7]; the last term in (15) can be as large as half the second (pressure-gradient) term. In unsteady flow without flow reversal, we use the same inner "boundary" conditions at $y_1 = 50 v/u_{\tau}$, but because of the presence of the time-dependent term in (2), α^* becomes more complicated. If we again assume that (16) holds - remember that the turbulence structure of the inner layer is unlikely to be affected unless the external-stream frequency is very high - then (1) and (2) give

$$\tau = \tau_{W} + \int_{0}^{y_{1}} \frac{\partial u}{\partial t} dy + \frac{\partial p}{\partial x} y_{1} + \int_{0}^{y_{1}} \frac{\partial}{\partial x} (u^{2}) dy + uv \bigg|_{y=y_{1}}$$
 (17)

Integrating we can write

$$\tau = \tau_{W} + \frac{\partial}{\partial t} \int_{0}^{y_{1}} u dy - u(y_{1}) \frac{\partial y_{1}}{\partial t} + \frac{\partial}{\partial x} \int_{0}^{y_{1}} u^{2} dy - u^{2}(y_{1}) \frac{\partial y_{1}}{\partial x}$$

$$- u(y_{1}) \frac{\partial}{\partial x} \int_{0}^{y_{1}} u dy + u^{2}(y_{1}) \frac{\partial y_{1}}{\partial x}$$
(18a)

because

$$\frac{\partial}{\partial x} \int_{0}^{y_{1}^{+}} \frac{u}{u_{\tau}} dy^{+} = 0$$

We can also write (18a) as

$$\tau = \tau_{W} + \frac{\partial p}{\partial x} y_{1} + v \frac{\partial}{\partial t} \int_{0}^{y_{1}^{+}} \frac{u}{u_{\tau}} dy^{+} + u_{\tau} F(y_{1}^{+}) \frac{vy_{1}^{+}}{u_{\tau}^{2}} \frac{\partial u_{\tau}}{\partial t} + v \frac{\partial u_{\tau}}{\partial x} \int_{0}^{y_{1}^{+}} \left(\frac{u}{u_{\tau}}\right)^{2} dy^{+}$$
(18b)

or as

$$\tau_{1} = \tau_{W} + y_{1}F \frac{\partial u_{\tau}}{\partial t} + \alpha * y_{1} \frac{\partial \tau_{W}}{\partial x} + \frac{\partial p}{\partial x} y_{1}$$
 (18c)

where $F = u/u_{\tau}$ at $y = y_{1}$ and α^{*} comes from the last term in (18b) and is the same as in steady flow.

Equation (18c) now replaces (15).

III. SOLUTION PROCEDURE

We use Keller's two-point finite-difference method (called the Box method) to solve the system of equations described in the previous section. The application of this method to unsteady flows with no flow reversal using the CS model has been described in Ref. [8]. Its application to steady two-dimensional flows using the BF model is described in Ref. [7]. Here we present a description of the extension of the CS model to unsteady two-dimensional turbulent flows with flow reversal as well as a description of the extension of the BF model to unsteady turbulent flows with no flow reversal.

3.1 CS Method with and without Flow Reversal

As in previous studies (see, for example [8]), we transform the equations with

$$\overline{x} = x/L$$
, $\overline{t} = tu_0/L$, $\eta = (u_0/\sqrt{x})^{\frac{1}{2}}y$ (19a)

and a dimensionless stream function $f(x,\eta,\overline{t})$, where

$$\psi = \left(u_{0} \vee x\right)^{\frac{1}{2}} f(\overline{x}, \eta, \overline{t}) \tag{19b}$$

Here u_0 is a reference velocity, L a reference length, and ψ is the usual definition of the stream function corresponding to the continuity equation (1). With the relations defined by (19) and with the definition of eddy viscosity, equations (1) to (3) and the boundary conditions can be written as

$$(bf'')' + \frac{1}{2}ff'' + m_3 = \overline{x}\left(f' \frac{\partial f'}{\partial \overline{x}} - f'' \frac{\partial f}{\partial \overline{x}} + \frac{\partial f'}{\partial \overline{t}}\right)$$
 (20)

$$\eta = 0$$
, $f = f' = 0$; $\eta \to \eta_{\infty}$, $f' = u_e/u_0 = \overline{u}_e$ (21)

Primes denote differentiation with respect to n and

$$f' = u/u_{0}, p_{3} = \overline{x} \left(\overline{u}_{e} \frac{\partial \overline{u}_{e}}{\partial \overline{x}} + \frac{\partial \overline{u}_{e}}{\partial \overline{t}} \right)$$

$$b = 1 + \varepsilon_{m}^{+}, \varepsilon_{m}^{+} = \frac{\varepsilon_{m}}{v}$$
 (22)

For simplicity, we shall now drop the bars on x and t.

We use two separate solution procedures to solve the system given by Eqs. (20) and (21). When there is no flow reversal across the layer, we use the standard Box. On the other hand, when there is flow reversal, then we use the so-called zig-zag Box as described below.

To solve Eqs. (20) and (21) by the standard Box method, we first write Eq. (20) in terms of three first-order equations by introducing new dependent variables u(x,n,t), v(x,n,t), that is,

$$f' = u \tag{23a}$$

$$u' = v \tag{23b}$$

$$(bv)' + \frac{1}{2} fv + P_3 = x(u \frac{\partial u}{\partial x} - v \frac{\partial f}{\partial x} + \frac{\partial u}{\partial t})$$
 (23c)

We next consider the net cube shown in Fig. 1 and write difference approximations to Eqs. (23). Equations (23a,b) are approximated using centered difference quotients and averaged about the midpoint $(x_i, t_n, n_{j-\frac{1}{2}})$. The difference quotients which are to approximate (23c) are written about the midpoint $(x_{i-\frac{1}{2}}, t_{n-\frac{1}{2}}, n_{j-\frac{1}{2}})$ of the cube whose mesh widths are r_i, k_n , and h_i . This procedure yields the following equations:

$$f_{j}^{i,n} - f_{j-1}^{i,n} - h_{j}u_{j-\frac{1}{2}}^{i,n} = 0$$
 (24a)

$$u_{j}^{i,n} - u_{j-1}^{i,n} - h_{j}v_{j-2}^{i,n} = 0$$
 (24b)

$$\frac{(bv)_{j}^{i,n} - (bv)_{j-1}^{i,n}}{h_{j}} + \frac{1}{2} (fv)_{j-\frac{1}{2}}^{i,n} - \alpha_{i} (u^{2})_{j-\frac{1}{2}}^{i,n} + \frac{\alpha_{i}}{2} (v_{j-\frac{1}{2}}^{i,n} f_{j-\frac{1}{2}}^{i,n} + m_{3} f_{j-\frac{1}{2}}^{i,n} + m_{4} v_{j-\frac{1}{2}}^{i,n}) - 2\beta_{n} u_{j-\frac{1}{2}}^{i,n} = n_{3}$$
(24c)

where

$$\alpha_{i} = \frac{x_{i-\frac{1}{2}}}{x_{i} - x_{i-1}}, \quad \beta_{n} = \frac{x_{i-\frac{1}{2}}}{t_{n} - t_{n-1}}, \quad m_{3} = v_{j-\frac{1}{2}}^{234}, \quad m_{4} = f_{j-\frac{1}{2}}^{(4)} - 2\overline{f}_{i-1}$$

$$n_{3} = \alpha_{i} [(u^{2})_{j-\frac{1}{2}}^{(4)} - 2(\overline{u^{2}})_{i-1}] - \frac{\alpha_{i}}{2} m_{3} m_{4} + 2\beta_{n} [u_{j-\frac{1}{2}}^{(2)} - 2\overline{u}_{n-1}] - h_{j}^{-1} [(bv)_{j}^{234} - (bv)_{j-1}^{234}] - \frac{1}{2} (fv)_{j-\frac{1}{2}}^{234} - 4(P_{3})_{n-\frac{1}{2}}^{i-\frac{1}{2}}$$

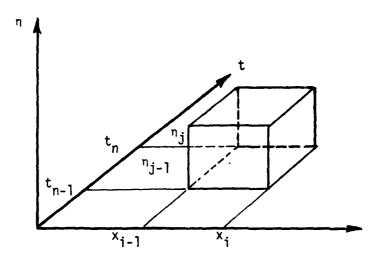


Fig. 1 Net cube for the standard Box method.

Here by v_j^{234} we mean $v_j^{i-1,n} + v_j^{i-1,n-1} + v_j^{i,n-1}$, the sum of the values of v_j at three of the four corners of the face of the box. Also

$$\overline{f}_{i-1} = \frac{1}{2} (f_{j-\frac{1}{2}}^{n,i-1} + f_{j-\frac{1}{2}}^{n-1,i-1})$$

$$\overline{u}_{n-1} = \frac{1}{2} (u_{j-\frac{1}{2}}^{n-1,i} + u_{j-\frac{1}{2}}^{n-1,i-1})$$

$$f_{j-\frac{1}{2}}^{i,n} = \frac{1}{2} (f_{j}^{i,n} + f_{j-1}^{i,n})$$

The resulting algebraic system given by Eqs. (24) together with the boundary conditions, which now become

$$f_0 = u_0 = 0, \qquad u_0 = \overline{u}_e \tag{25}$$

are nonlinear. We use Newton's method to linearize the system and solve the linear system by the block elimination method discussed in ref. [9].

When there is flow reversal across the boundary layer at some x and t, we modify the standard Box method used for Eq. (23c) but retain that for (23a,b) and still center them at $(x_i, t_n, n_{j-\frac{1}{2}})$. To write the difference approximations for the Box centered at $(x_{i,1}, t_n, n_{j-\frac{1}{2}})$ we examine previously computed values of $u_{j-\frac{1}{2}}^{i,n}$. If $u_{j-\frac{1}{2}}^{i,n} \geq 0$, then we use the standard Box method: if $u_{j-\frac{1}{2}}^{i,n} < 0$, then we write (23c) for the Box centered at P (see Fig. 2) using quantities centered at P, Q, and R, where

$$P = (x_{1}, t_{n-\frac{1}{2}}, \eta_{j-\frac{1}{2}}), \qquad Q = (x_{1-\frac{1}{2}}, t_{n}, \eta_{j-\frac{1}{2}})$$

$$R = (x_{1+\frac{1}{2}}, t_{n-1}, \eta_{1-\frac{1}{2}}) \qquad (26)$$

Equation (23c) can then be written as

$$(bv)'(P) + \frac{1}{2} (fv)(P) = x(P) \left[eu(Q) \frac{\partial u}{\partial x} (Q) + \phi u(R) \frac{\partial u}{\partial x} (R) - ev(Q) \frac{\partial f}{\partial x} (Q) - \phi v(R) \frac{\partial f}{\partial x} (R) + \frac{\partial u}{\partial t} (P) \right]$$
(27)

Here

$$\theta = \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}}, \qquad \phi = \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}}$$
 (28)

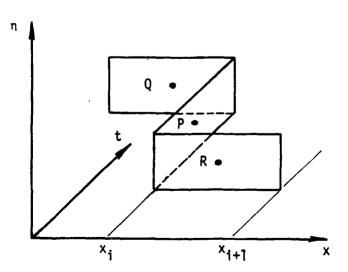


Fig. 2 Finite difference molecule for the Zig-Zag differencing.

The resulting algebraic system is again nonlinear and its solution is obtained by using the procedure followed in the standard Box method.

3.2 BF Method with no Flow Reversal

The solution of the governing equations for unsteady flows with the BF model, even with no flow reversal across the boundary layer, is much more difficult than with the CS model. This is because of the hyperbolic nature of the governing equations, together with the nonlinear boundary conditions, which play an important role in the solution procedure. As is common in most (if not all) methods that use boundary conditions away from the "wall," the wall shear stress is also an unknown parameter; it can be treated as an eigenvalue or as a mechul as described in Ref. [7]. The latter procedure is much more efficient than the former procedure and is used here.

To solve the BF model equations, we first introduce the stream function $\psi(x,y)$ as in Ref. [7] in order to satisfy the continuity equation. With $\sqrt{\tau_W} \approx w$ treated as mechal, the resulting system can be written as a system of four first-order equations:

$$w' = 0 (29a)$$

$$\psi' = u \tag{29b}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - u' \frac{\partial \psi}{\partial x} = P_3 + \tau'$$
 (29c)

$$\frac{\partial \tau}{\partial t} + u \frac{\partial \tau}{\partial x} - \tau' \frac{\partial \psi}{\partial x} = 2a_1 \left[\tau u' - \frac{\tau^{3/2}}{L} - \tau_{\text{max}}^{\frac{1}{2}} (G\tau)' \right]$$
 (29d)

We again center Eqs. (29a,b) about the midpoint $(x_i, t_n, \eta_{j-\frac{1}{2}})$ and Eqs. (29c,d) about the midpoint $(x_{i-\frac{1}{2}}, t_{n-\frac{1}{2}}, \eta_{j-\frac{1}{2}})$ of the cube shown in Fig. 1. This procedure yields the following nonlinear algebraic equations:

$$w_{j}^{i,n} - w_{j-1}^{i,n} = 0$$
 (30a)

$$\psi_{j}^{i,n} - \psi_{j-1}^{i,n} - h_{j}u_{j-1}^{i,n} = 0$$
 (30b)

$$\frac{\tau_{j}^{i,n} - \tau_{j-1}^{i,n}}{h_{j}} + \alpha_{i} \left[\frac{u_{j-1}^{i,n} - u_{j-1}^{i,n}}{h_{j}} (\psi_{j-1_{2}}^{i,n} - \psi_{j-1_{2}}^{i-1,n}) + \psi_{j-1_{2}}^{i,n} \frac{u_{j-1}^{i-1,n} - u_{j-1}^{i,n}}{h_{j}} - (u^{2})_{j-1_{2}}^{i,n} \right] - 2\beta_{n} u_{j-1_{2}}^{i,n} = n_{3}$$
 (30c)

$$\beta_{n}\tau_{j-i_{2}}^{i,n} + \widehat{\alpha}_{i} \left[u_{j-i_{2}}^{i,n} \left(\tau_{j-i_{2}}^{i,n} - \tau_{j-i_{2}}^{i-1,n} \right) + u_{j-i_{2}}^{i-1,n} \tau_{j-i_{2}}^{i,n} \right] - \widehat{\alpha}_{i} \left[\frac{\tau_{j}^{i,n} - \tau_{j-1}^{i,n}}{h_{j}} \left(\psi_{j-i_{2}}^{i,n} - \psi_{j-i_{2}}^{i-1,n} \right) + u_{j-i_{2}}^{i-1,n} \tau_{j-i_{2}}^{i,n} \right] - \widehat{\alpha}_{i} \left[\frac{\tau_{j}^{i,n} - \tau_{j-1}^{i,n}}{h_{j}} \left(\psi_{j-i_{2}}^{i,n} - \psi_{j-i_{2}}^{i-1,n} \right) \right] - \tau_{j-i_{2}}^{i,n} \frac{u_{j}^{i,n} - u_{j-1}^{i,n}}{h_{j}} + 2 \left[\frac{(\tau_{j}^{3/2})_{j-i_{2}}^{i,n} + (\tau_{j-i_{2}}^{3/2})_{j-i_{2}}^{i-1,n}}{L_{j-i_{2}}^{i,n} + L_{j-i_{2}}^{i-1,n}} \right] + (G^{i})_{j-i_{2}}^{i,n} \tau_{j-i_{2}}^{i,n} + G_{j-i_{2}}^{i,n} \frac{\tau_{j}^{i,n} - \tau_{j-1}^{i,n}}{h_{j}} = n_{4}$$
(30d)

where now

$$\begin{split} \alpha_{i} &= \frac{1}{x_{i} - x_{i-1}}, \qquad \beta_{i} &= \frac{1}{t_{n} - t_{n-1}}, \qquad \widetilde{\alpha}_{i} &= \frac{\alpha_{i}}{2a_{1}}, \qquad \widetilde{\beta}_{n} &= \frac{\beta_{n}}{2a_{1}} \\ n_{3} &= -4(P_{3})_{n-\frac{1}{2}}^{i-\frac{1}{2}} - (\tau')_{j-\frac{1}{2}}^{234} + 2\beta_{n}(u_{j-\frac{1}{2}}^{i-1}, n - u_{j-\frac{1}{2}}^{i-1}, n-1) - u_{j-\frac{1}{2}}^{i,n-1}) + \alpha_{i}[(u^{2})_{j-\frac{1}{2}}^{i,n-1} \\ &- (u^{2})_{j-\frac{1}{2}}^{i-1}, n-1 - (u^{2})_{j-\frac{1}{2}}^{i-1}, n] - \alpha_{i} \left[\{(u')_{j-\frac{1}{2}}^{i,n-1} + (u')_{j-\frac{1}{2}}^{i-1}, n-1 \} \\ &(\psi_{j-\frac{1}{2}}^{i,n-1} - \psi_{j-\frac{1}{2}}^{i-1}, n-1) - (u')_{j-\frac{1}{2}}^{i-1}, n\psi_{j-\frac{1}{2}}^{i-1}, n \right] \\ n_{4} &= \widetilde{\alpha}_{i}(A_{3} - A_{2}) - 2\beta_{n}A_{1} + (\tau u')_{j-\frac{1}{2}}^{234} - 2A_{4} - (G'\tau)_{j-\frac{1}{2}}^{234} - (G\tau')_{j-\frac{1}{2}}^{234} \\ A_{1} &= \tau_{j-\frac{1}{2}}^{i-1}, n - \tau_{j-\frac{1}{2}}^{i-1}, n-1 - \tau_{j-\frac{1}{2}}^{i-1}, \\ A_{2} &= 2u_{j-\frac{1}{2}}^{i-\frac{1}{2}}, n-1 (\tau_{j-\frac{1}{2}}^{i,n-1} - \tau_{j-\frac{1}{2}}^{i-1}, n-1) - u_{j-\frac{1}{2}}^{i-1}, n^{\tau_{j-\frac{1}{2}}}, \\ A_{3} &= [(\tau')_{j-\frac{1}{2}}^{i,n-1} + (\tau')_{j-\frac{1}{2}}^{i-1}, n-1] [\psi_{j-\frac{1}{2}}^{i,n-1} - \psi_{j-\frac{1}{2}}^{i-1}, n-1] - (\tau')_{j-\frac{1}{2}}^{i-1}, n \psi_{j-\frac{1}{2}}^{i-1}, n \\ A_{4} &= \frac{(\tau^{3/2})_{j-\frac{1}{2}}^{i,n-1} + (\tau^{3/2})_{j-\frac{1}{2}}^{i-1}, n-1}{L_{j-\frac{1}{2}}^{i-1}, n-1} + L_{j-\frac{1}{2}}^{i-1}, n-1} \end{aligned}$$

Again the system given by Eqs. (30) is nonlinear and is linearized by using Newton's method. This procedure gives rise to the following form (2 < j < J)

$$\delta w_{j} - \delta w_{j-1} = (r_{3})_{j}$$
 (31a)

$$\delta \psi_{j} - \delta \psi_{j-1} - \frac{n_{j}}{2} (\delta u_{j} + \delta u_{j-1}) = (r_{4})_{j-1}$$
 (31b)

$$(s_{1})_{j}\delta u_{j} + (s_{2})_{j}\delta u_{j-1} + (s_{3})_{j}\delta \psi_{j} + (s_{4})_{j}\delta \psi_{j-1} + (s_{5})_{j}\delta \tau_{j} + (s_{6})_{j}\delta \tau_{j-1} = (r_{1})_{j}$$

$$(31c)$$

$$(\beta_{1})_{j}\delta u_{j} + (\beta_{2})_{j}\delta u_{j-1} + (\beta_{3})\delta \psi_{j} + (\beta_{4})_{j}\delta \psi_{j-1} + (\beta_{5})_{j}\delta \tau_{j} + (\beta_{6})_{j}\delta \tau_{j-1} = (r_{2})_{j}$$

$$(31d)$$

Here for convenience we have dropped the superscripts i,n and have defined $(s_k)_j$ (k=1, 2, ..., 6)

$$(s_{1})_{j} = -\beta_{n} - \alpha_{i}u_{j} + \alpha_{i}/h_{j} \quad (\psi_{j-i_{2}} - \psi_{j-i_{2}}^{i-1},^{n})$$

$$(s_{2})_{j} = -\beta_{n} - \alpha_{i}u_{j} - \alpha_{i}/h_{j} \quad (\psi_{j-i_{2}} - \psi_{j-i_{2}}^{i-1},^{n})$$

$$(s_{3})_{j} = \alpha_{i}/2 \quad [(u')_{j-i_{2}} + (u')_{j-i_{2}}^{i-1},^{n}],$$

$$(s_{4})_{j} = (s_{3})_{j}$$

$$(s_{5})_{j} = 1/h_{j}, \qquad (s_{6})_{j} = -1/h_{j}$$
and
$$(\beta_{k})_{j} \quad (k = 1, 2, \dots, 6)$$

$$(\beta_{1})_{j} = \frac{\alpha_{i}}{2} \left(\tau_{j-i_{2}} - \tau_{j-i_{2}}^{i-1},^{n}\right) - \frac{1}{h_{j}} \tau_{j-i_{2}}$$

$$(\beta_{2})_{j} = \frac{\alpha_{i}}{2} \left(\tau_{j-i_{2}} - \tau_{j-i_{2}}^{i-1},^{n}\right) + \frac{1}{h_{j}} \tau_{j-i_{2}}$$

$$(\beta_{3})_{j} = -\frac{\alpha_{i}}{2} \left[(\tau')_{j-i_{2}} + (\tau')_{j-i_{2}}^{i-1},^{n}\right], \quad (\beta_{4})_{j} = (\beta_{3})_{j}$$

$$+ \frac{1}{2} [(G')_{j-\frac{1}{2}} - (u')_{j-\frac{1}{2}}] + \frac{3}{2} \frac{\sqrt{|\tau_{j}|}}{\frac{1-1}{1-1}, n + L_{j-\frac{1}{2}}} + \frac{G_{j-\frac{1}{2}}}{h_{j}}$$

$$(\beta_{6})_{j} = \beta_{n} + \frac{\tilde{\alpha}_{j}}{2} (u_{j-\frac{1}{2}} - u_{j-\frac{1}{2}}^{i-1}, n) - \frac{\tilde{\alpha}_{j}}{h_{j}} (\psi_{j-\frac{1}{2}}^{i-1}, n - \psi_{j-\frac{1}{2}})$$

$$+ \frac{1}{2} [(G')_{j-\frac{1}{2}} - (u')_{j-\frac{1}{2}}] + \frac{3}{2} \frac{\sqrt{|\tau_{j}|}}{\frac{1-1}{1-1}, n + L_{j-\frac{1}{2}}} - \frac{G_{j-\frac{1}{2}}}{h_{j}}$$

 $(\beta_5)_{j} = \beta_n + \frac{\alpha_j}{2} (u_{j-\frac{1}{2}} - u_{j-\frac{1}{2}}^{j-1}, n) + \frac{\alpha_j}{h_i} (\psi_{j-\frac{1}{2}}^{j-1}, n - \psi_{j-\frac{1}{2}})$

The terms denoted by $(r_k)_1$ (k = 1, 2, 3, 4) are defined by:

$$(r_{3})_{j} = 0$$

$$(r_{4})_{j-1} = \psi_{j-1} - \psi_{j} + h_{j}u_{j-i_{2}}$$

$$(r_{1})_{j} = n_{3} - [2\beta_{n}u_{j-i_{2}} + \alpha_{1}(u^{2})_{j-i_{2}} - \alpha_{1} \{(u')_{j-i_{2}}(\psi_{j-i_{2}} - \psi_{j-i_{2}}^{i-1}, n)$$

$$+ (u')_{j-i_{2}}^{i-1}, n_{\psi_{j-i_{2}}} \} - (\tau')_{j-i_{2}}]$$

$$(r_{2})_{j} = n_{4} - \left[2\beta_{n}\tau_{j-i_{2}} + \alpha_{1}^{i} \{(u_{j-i_{2}}(\tau_{j-i_{2}} - \tau_{j-i_{2}}^{i-1}, n) + u_{j-i_{2}}^{i-1}, n_{\tau_{j-i_{2}}} \}$$

$$- \alpha_{1}^{i} \{(\tau')_{j-i_{2}}(\psi_{j-i_{2}} - \psi_{j-i_{2}}^{i-1}, n) + (\tau')_{j-i_{2}}^{i-1}, n_{\psi_{j-i_{2}}} \} - \tau_{j-i_{2}}(u')_{j-i_{2}}$$

$$+ 2 \left\{ \frac{\tau_{3}^{3/2} + (\tau_{3}^{3/2})_{j-i_{2}}^{i-1}, n}{L_{j-i_{2}}} \right\} + G_{j-i_{2}}^{i}\tau_{j-i_{2}}^{i-1} + G_{j-i_{2}}^{i}(\tau')_{j-i_{2}} \right]$$

For j = 1, we use the boundary conditions given by Eqs. (13), (14) and (15) and first write them as:

$$u_{1} = w_{1}(2.5 \text{ In } \frac{y_{1}w_{1}}{v} + 5.2)$$

$$\frac{\partial \psi_{1}}{\partial x} - \frac{u_{1}y_{1}}{w_{1}} \frac{\partial w_{1}}{\partial x} = 0$$

$$\tau_{1} = w_{1}^{2} + y_{1} \frac{u_{1}}{w_{1}} \frac{\partial w_{1}}{\partial t} + \alpha * y_{1} \frac{\partial}{\partial x} w_{1}^{2} - P_{3}y_{1}$$

After we write the difference equations and linearize the resulting nonlinear expressions, we get

$$\delta u_1 - \left[2.5(\ln \frac{y_1 w_1}{v} + \frac{v}{y_1}) \right] \delta w_1 = (r_1)_1$$
 (32a)

$$y_1(w_1 - E_2)\delta u_1 + (w_1 + w_1^{234})\delta \psi_1 + [(\psi_1 - E_1) - y_1(u_1 - u_1^{234})]\delta w_1 = (r_2)_1$$
(32b)

$$\delta \tau_1 + g_7 \delta w_1 = (r_3)_1$$
 (32c)

where

$$E_{1} = \psi_{1}^{i-1}, n - \psi_{1}^{i}, n-1 + \psi_{1}^{i-1}, n-1$$

$$E_{2} = (w_{1}^{2})^{i-1}, n - (w_{1}^{2})^{i}, n-1 + (w_{1}^{2})^{i-1}, n-1$$

$$E_{4} = -w_{1}^{i}, n-1 + w_{1}^{i-1}, n - w_{1}^{i-1}, n-1$$

$$E_{5} = \frac{1}{2y_{1}^{+}} \int_{0}^{y_{1}^{+}} [2.5 \ln(1.0 + y_{1}^{+}) + 5.1 - (3.39y_{1}^{+} + 5.1) \exp(-0.37y_{1}^{+})]^{2} dy_{1}^{+}$$

$$g_{7} = -2w_{1} \left[1 + \frac{1}{2} y_{1}\alpha_{1}E_{5}^{1234}\right] - \frac{1}{2} y_{1}\beta_{n} \left(\frac{u_{1}}{w_{1}}\right)^{1234}$$

$$(r_{1})_{1} = w_{1}[2.5 \ln \frac{y_{1}w_{1}}{v} + 5.2] - u_{1}$$

$$(r_{2})_{1} = y_{1}u_{1}^{1234}(w_{1} - E_{2}) - w_{1}^{1234}(\psi_{1} - E_{1})$$

$$(r_{3})_{1} = (w_{1}^{2})^{234} - \tau_{1}^{234} - 4(P_{3})_{5-\frac{1}{2}y_{1}}^{n-\frac{1}{2}y_{1}} + \frac{1}{2} y_{1}\beta_{n} \left(\frac{u_{1}}{w_{1}}\right)^{1234} E_{4} + \frac{1}{2} y_{1}\alpha_{1}E_{5}^{1234}E_{3}$$

$$- (\tau_{1} - w_{1}^{2} - \frac{1}{2} y_{1}\beta_{n} \left(\frac{u_{1}}{w_{1}}\right)^{1234} w_{1} - \frac{1}{2} y_{1}\alpha_{1}E_{5}^{1234}w_{1}^{2}]$$

For j = J, we use the usual boundary condition,

which in its linearized form is

$$\delta u_{\rm J} = (r_4)_1 = 0$$
 (33)

The equations (31) for $2 \le j < J$ and the boundary conditions given by Eqs. (32) and (33) form a linear system which is solved by the blockelimination method discussed in Ref. [9].

IV. RESULTS AND DISCUSSION

To study the calculation of unsteady turbulent boundary layers with and without flow reversal we have considered three separate test cases. The first one has an external velocity distribution of the form

$$\overline{u}_{p} = 1 - \alpha(x - x^{2})(t^{2} - t^{3})$$
 0 < x < 1, t > 0 (34)

where α is a positive constant. The same velocity distribution was recently used by Cebeci [1] for laminar flows in order to study the computation of unsteady laminar flows with flow reversal using the solution procedure described in the previous section and to see whether there is a singularity associated with such flows.

In performing calculations for this case and for the others considered here, care must be taken in generating the initial conditions in the (t,y) and (x,y) planes at some distance, say $x=x_0$. For a laminar flow if $x_0=0$, the initial velocity profile for the velocity distribution given by Eq. (34) can be taken as Blasius and there is no difficulty about computing the solution in x>0 since the initial boundary layer is of zero thickness. If $x_0 \neq 0$, we can take

$$\overline{u}_e = 1 - \alpha(x_0 - x_0^2)(t^2 - t^3)$$
 $0 < x < x_0$

but then at $x = x_0$ there is a discontinuity in the pressure gradient. Since it acts on an already-established boundary layer, the initial response is inviscid leading formally to a velocity slip and hence a subboundary layer at the wall. The treatment of the boundary layer is then rather subtle (see Ref. [10]) but if we are not too concerned with the details of the solution near $x = x_0$, which is the case here, a convenient procedure would be to write Eq. (34) as

$$\overline{u}_{e} = 1 - \alpha F[(x - x_{0})/a](x - x^{2})(t^{2} - t^{3})$$
 (35)

where F is a smooth function which vanishes if $x < x_0$ and is unity if $x - x_0 > a$. For example, we can take $F(s) = \sin(\pi s/2) = 0 < s < 1$, and a = 0.06 with ten stations between $x = x_0$ and $x = x_0 + a$. A similar difficulty would occur at t = 0 if t^2 in Eq. (34) were replaced by t since the boundary layer is well established at t = 0.

Figures 3 and 4 show the results for the turbulent flow calculations with the CS model for the test case given by Eq. (35) with α = 40 and a unit Reynolds number $u_0/v = 2.2 \times 10^6/m$. The results shown in Fig. 3 were obtained by using different expressions for A; those shown by circles were obtained with Eq. (7), and those shown by solid lines with Eq. (7) written as

$$A = 26(\frac{\tau}{\rho})^{-\frac{1}{2}}$$
 (36)

As can be seen, both expressions give nearly the same results.

The results in Fig. 4, as in laminar flows, exhibit no signs of singularity for all calculated values of t. This is in contrast to the findings of Patel and Nash [11]. Again, as in laminar flows [3], we see the familiar rapid thickening of the boundary layer in the reversed flow region. If it had not been for this, the calculations would have been computed for greater values of t than those considered here.

The two other test cases considered here correspond to Cases 4 and 5, as reported by Carr [12]. Case 4 is for unsteady Howarth flow. It starts from a well-established steady flat-plate flow, on which a linear deceleration of u_e is imposed at t=0. The external velocity distribution is given by

where $\overline{\alpha}$ is a constant equal to 2.4/3.45 sec⁻¹m⁻¹. The flow was assumed to be steady up to x = 1.24m; the velocity distribution Eq. (37) was then imposed as a function of x and t. This test case differs from the previous one in that, once the flow separates, it does not reattach. For this reason, the calculations can only be continued as far as the station where the flow reversal first occurs. The initial velocity profile at x = 1.24 and for all time correspond to a flat-plate profile with a momentum thickness Reynolds number (R_{θ}) of 4860, and local skin-friction coefficient c_f of 2.8 x 10^{-3} .

As in the previous test case, we introduce a function F_1 so that at x = 1.24, $du_e/dx = 0$. Since we also want the solutions at t = 0 to correspond to steady-state solutions, we introduce another function F_2 in order to set $\partial u_e/\partial t = 0$. With these functions, Eq. (37) then becomes

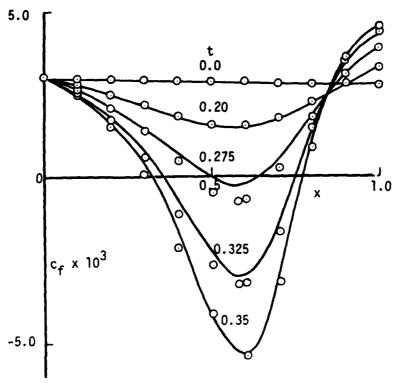


Figure 3. Local skin-friction variation with $\,x\,$ for various values of $\,z\,$. Solid lines denote the calculations made by (29) and circles by (8).

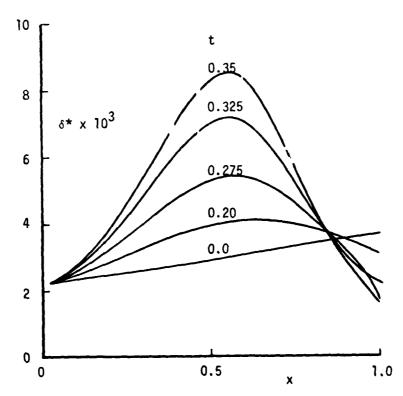


Figure 4. Variation of displacement thickness with \times for various values of t.

$$\overline{u}_{e} = 1 - \overline{\alpha}F_{1}F_{2}(x - 1.24)t$$
 (38)

where

$$F_1 = \sin \frac{\pi}{2} \left(\frac{x - 1.24}{0.1} \right), \qquad F_2 = \sin \frac{\pi}{2} \frac{t}{1.98}$$

Figures 5, 6 and 7 show the calculated local skin-friction coefficient $c_{\bf f},$ the shape factor H and the momentum thickness Reynolds number R_{θ} for this test case. The calculations were done by using both CS and BF models; the results shown by solid lines refer to the predictions of the CS model and those shown by circles refer to the predictions of the BF model.

As seen from these three figures, there is essentially no difference between the predictions of both models. Although there is some discrepancy in the shape factor predictions, this does not seem to be too significant.

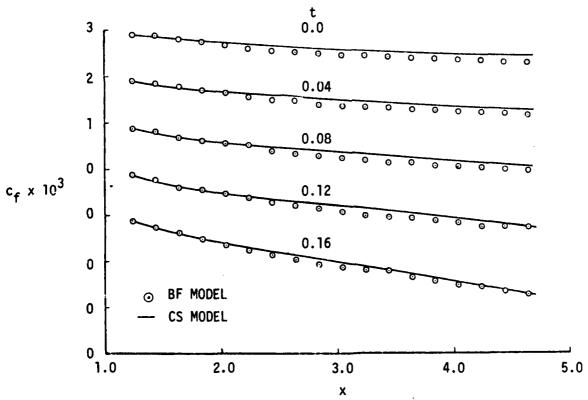


Figure 5. Computed local skin-friction distribution for test case 4.

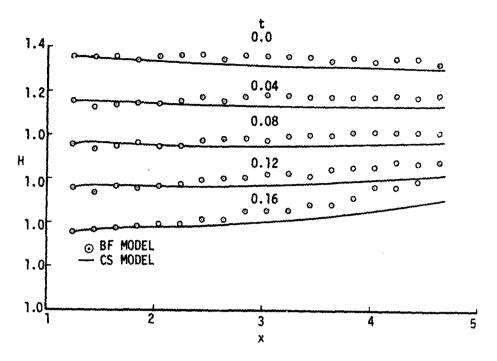


Figure 6. Computed shape factor distribution for test case 4.

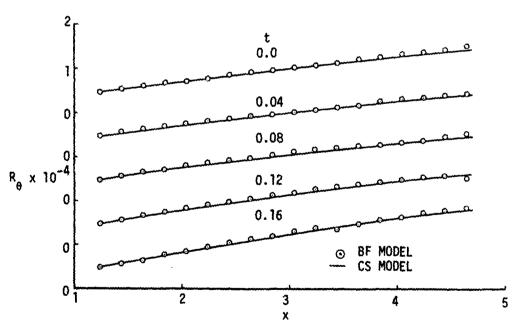


Figure 7. Computed momentum-thickness Reynolds number for test case 4.

According to the predictions of the CS model, which also has the capability of predicting unsteady boundary layers with flow reversal, the wall shear vanishes first around t \sim 0.22, x = 4.69. Since the computation of boundary layers for values of x in the range $1.24 \le x \le 4.69$ for t > 0.22 depends on the specification of a velocity profile at x = 4.69, we generate such a profile by assuming it is given by the extrapolation of two velocity profiles computed for x < 4.69. This procedure in which the extrapolated station serves as a downstream boundary condition, allows the calculations to be continued in the negative wall shear region as shown in Fig. 8.

The third case considered in our study corresponds to Case 5 in ref. [12], which in a way resembles the external velocity distribution in Eq. (34). It is given by

$$\overline{u}_{e} = 1 + \left\{A^{2} + (Bt)^{2} \left[\xi - \xi_{0}\right]^{2}\right\}^{\frac{1}{2}} - \left[A^{2} + (B\xi_{0}t)^{2}\right]^{\frac{1}{2}}$$
 (39)

where A = 0.05, B = 3.4 sec⁻¹, ξ = (x - 1.24)/3.45 and the range of x values are limited to $1.24 \le x \le 4.69$. As before, the initial velocity profiles at x = 1.24 for all t correspond to a steady flat-plate flow with R_{θ} = 4860, c_f = 2.89 x 10^{-3} . We again modify Eq. (39) to avoid the discontinuity in the pressure gradient. This time we multiply the right-hand side of Eq. (39) by F_1 used in Eq. (38).

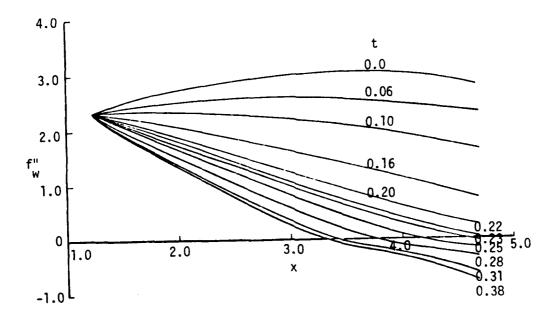


Figure 8. Variation of wall-shear parameter $f_W^{\prime\prime}$ with distance as a function of time for test case 4.

Figures 9 and 10 show the calculated local skin-friction coefficient c_{f} and the momentum thickness Reynolds number $R_{\mathrm{\theta}}$ for this test case. Again we present the predictions of both turbulence models. Figure 11 shows the calculated velocity profiles for several t and x-stations. As is seen from these figures, the predictions of both turbulence models are the same for all practical purposes.

Figure 12 shows the variation of wall shear parameter f_W^* as a function of x and t, and Figure 13 shows the calculated velocity profiles, including the regions in which there is flow reversal across the boundary layer. These computations which are done by using the CS model provide confirmation of the general trends in test case 4, namely that as in laminar flows, the unsteady turbulent boundary layers thicken rapidly with increasing flow reversal. A new feature however is the dip in the graphs of f_W^* near x=2.5 which develops as t increases towards 0.40. It is possible that a singularity occurs in the solution at a later time as many authors have suggested is the case for laminar boundary layers. The most cogent argument in favor of this phenomenon has been advanced by Shen [13] but we note that the most definite sign of its occurrence appeared in his graphs of displacement thickness which showed spikey characteristics. Here the displacement thickness seems to be fairly smooth but the skin friction becomes spikey.

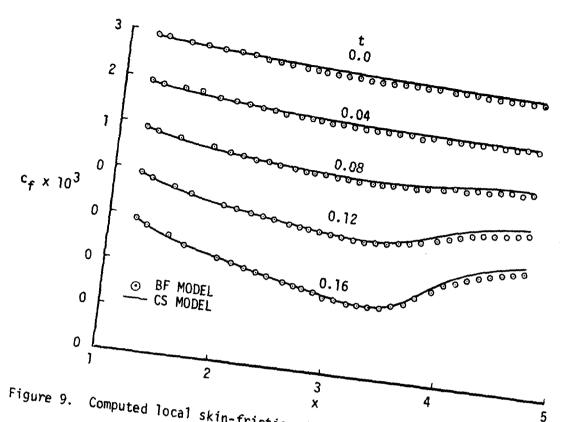


Figure 9. Computed local skin-friction distribution for test case 5.

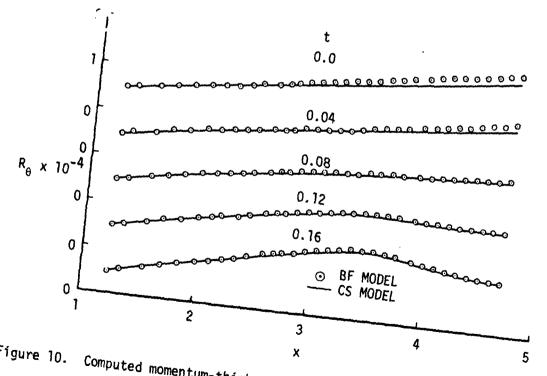


Figure 10. Computed momentum-thickness Reynolds number for test case 5. 22

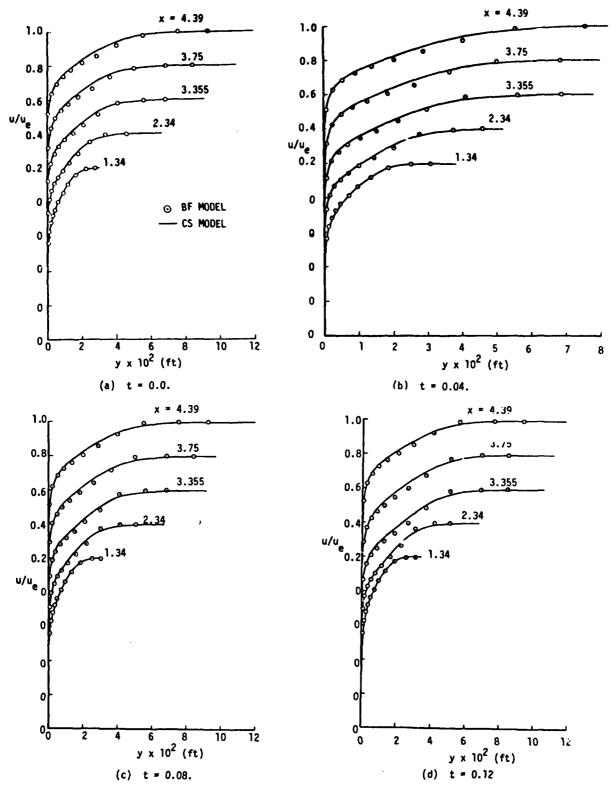


Figure 11. Comparison of calculated velocity profiles for test case 5 with no flow reversal.

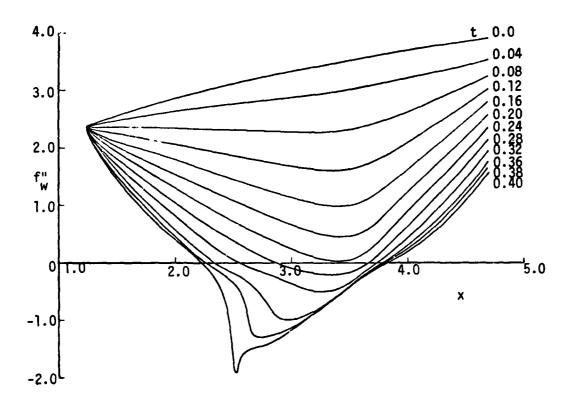


Figure 12. Variation of wall shear parameter $\mbox{ f}_{w}^{"}$ with distance as a function of time for test case 5.

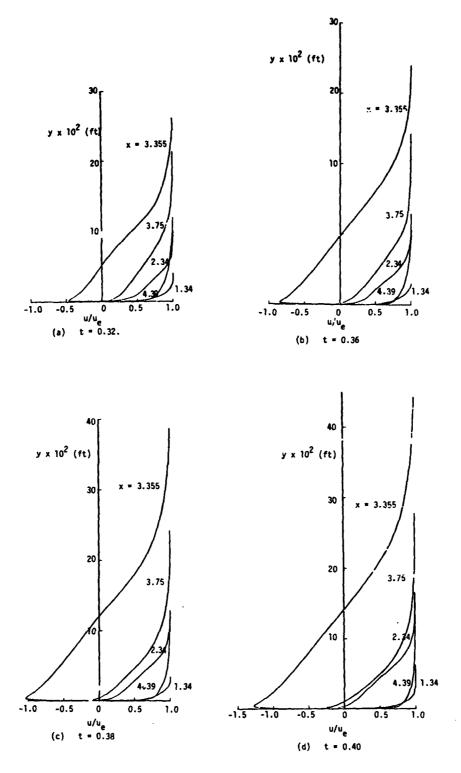


Figure 13. Calculated velocity profiles including flow reversal by the CS models for test case 5.

V. CONCLUDING REMARKS

Based on the studies conducted in this report, we observe that:

- 1. The numerical solution of unsteady laminar and turbulent boundary layers including the flow reversal across the layer can be obtained quite satisfactorily for a given pressure distribution. A combination of both regular and zig-zag box schemes are shown to yield accurate results for unsteady boundary layers.
- 2. Whether the unsteady boundary-layer equations for laminar and turbulent flows are singular for a given pressure distribution still remains to be investigated. The results for test case 5 indicate that at large times there is a puzzling "kink" in the wall shear parameter, $f_{\boldsymbol{w}}^{\prime\prime};$ this may be due to a singularity or it may be due to a numerical problem. Recent studies conducted by Cebeci [14] and van Dommelen and Shen [15] for a circular cylinder started impulsively from rest indicate that at large times, t = 1.25 or more, there appears to be a singularity in δ^* around $\phi = 120^{\circ}$. However, these calculations do not indicate any puzzling behavior in the wall shear parameter near "singularity;" the $f_{\omega}^{"}$ -values are smooth and well behaved for these and larger times. On the other hand, examining the δ^* -results for test case 5, we find that while there is an abnormal behavior in f_{ω}^{*} at large times, the corresponding 6*-values are smooth and well behaved, a trend which is opposite to that for a circular cylinder.
- 3. A comparison of the predictions of two turbulence models, namely, CS and BF models indicate that for attached flows, both models yield almost identical results. This is also true for flows which are sufficiently strong in pressure gradient to cause flow reversal across the layer.

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VII. DESCRIPTION OF THE COMPUTER PROGRAM WHICH USES THE CS MODEL

Input

Essentially the input to the computer program consists of four types of cards. Card I contains the title of the flow problem under consideration. Card 2 requires the following information to be specified.

NXT	Total number of t-stations to be calculated
NZT	Total number of x-stations to be calculated
NTR	x-station where transition begins. If the initial velocity profile is for turbulent flows, then NTR=1. If flow is all laminar, set NTR>NZT.
IBDY	Specifies whether the flow at x=0 starts as a flat-plate flow or as a stagnation-point flow. =1 flat-plate flow =2 stagnation-point flow
RL	Free-stream Reynolds number, u_L/v.
IPRNT	Controls the print output
	=1 prints out only the boundary-layer parameters δ^*, θ , c_f , R_{s*} , R_{a} , H and external velocity distribution.

=2 prints out profiles as well as the boundary-layer parameters and external velocity field.

DETA(1) and VGP are the nonuniform grid parameters that control the spacing across the layer. The grid used in this report is a reometric progression with the property that the ratio of lengths of any two adjacent intervals is a constant; that is, $\Delta n_j = K\Delta n_{j-1}$. The distance to the j-th line is given by the following formula:

$$\Delta n_{j} = h_{1}(K^{j} - 1)/(K - 1) \quad K > 1$$

There are two parameters in this equation: h_1 , the length of the first step, and K, the ratio of two successive steps. The total number of points J can be calculated from the following formula:

$$J = \frac{\ln[1 + (K - 1)(n_e/h_1)]}{\ln K}$$

In the computer program which embodies the present solution method, h_1 and K are chosen with typical values, for moderate Reynolds numbers, of 0.01 and 1.3, respectively. In general, approximately 50 grid nodes across the boundary layer are sufficient to represent laminar and turbulent boundary-layer flows. The chosen values of h_1 and K must be such that the formula which generates the number og rid nodes according to a given or estimated n_e , i.e. Eq. () does not allow J to exceed 101. Figure 14 is provided, therefore, to provide guidance in the selection of J.

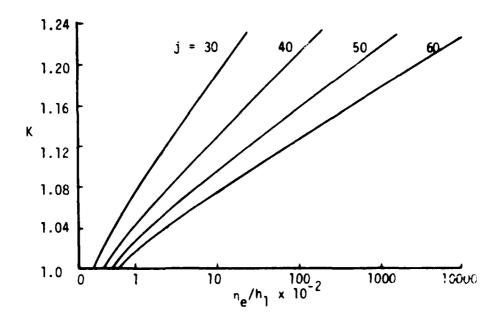


Figure 14. Variation of K with h_1 for different r_e -values.

CF and RTH are the local c_f and R_θ values which are used to start the turbulent flow calculations. The initial velocity profile is generated by using the formulas proposed by Granville (see ref. 9)

$$\frac{u}{u_{\tau}} = \frac{1}{\kappa} \left[\ln y^{+} + c + \pi (1 - \cos \pi n) + (n^{2} - n^{3}) \right]$$
 (40)

From Eq. (40) and from the definitions of δ^* and θ , it can be shown that

$$\frac{\delta^*}{\delta} = \int_0^1 \frac{u_e - u}{u_\tau} \frac{u_\tau}{u_e} d\eta = \frac{u_\tau}{\kappa u_e} (\frac{11}{12} + \pi)$$
 (41)

$$\frac{\theta}{\delta} = \int_{0}^{1} \frac{u}{u_{e}} \left(1 - \frac{u}{u_{e}}\right) d\eta = \frac{u_{\tau}}{\kappa u_{e}} \left(\frac{11}{12} + \pi\right) - \left(\frac{u_{\tau}}{\kappa u_{e}}\right)^{2} \left[2 + 2\pi(1 + \frac{1}{\pi} \operatorname{Si}(\pi)) + 1.5\pi^{2} + \frac{1}{105} - \frac{7}{72} - 0.122925\pi\right]$$
(42)

From Eq. (42), taking $Si(\pi) = 1.8519$, we can also write

$$\frac{R_{\theta}}{R_{\delta}} = \frac{u_{\tau}}{\kappa u_{e}} \left(\frac{11}{12} + \pi \right) - \left(\frac{u_{\tau}}{\kappa u_{e}} \right)^{2} (1.9123016 + 3.05603 + 1.5\pi^{2})$$

Evaluating Eq. (40) at $\eta = 1$, we get

$$\frac{u_e}{u_\tau} = \frac{1}{\kappa} \left[\ln \left(\frac{\delta u_e}{v} \frac{u_\tau}{u_e} \right) + c + 2\pi \right]$$
 (43)

For a given value of c_f and R_θ , we can solve Eqs. (42) and (43) for δ and π and then substitute them into Eqs. (40) and (41), thus obtaining u-profile.

Cards 3 and 4 read in the t and x stations, respectively. The present computer program specifies the external velocity distribution by a formula and computes the dimensionless pressure gradient parameters analytically as is shown in the listing which follows this section. The test case in the computer listing is for case 4 of ref. [12].

Output

Depending on the IPRNT, the computer program prints out the profiles f, f', f'' and b as a function of the similarity variable n and grid parameter j together with a parameter KALC(J). Here KALC(J) = 0 when we use the standard box and = 1 when we use the zig-zag box.

ETA
$$\eta$$

$$F \qquad f'$$

$$U \qquad f'$$

$$V \qquad f''$$

$$B \qquad b \ (=1 + \epsilon_m^+) \quad \text{equals 1.0 for laminar flows}$$

The output also includes displacement thickness δ^* , momentum thickness θ , local skin-friction coefficient $c_{\mathfrak{f}}$, Reynolds numbers based on δ^* and θ ,

that is, ${\rm R}_{\delta^{\bigstar}},\,{\rm R}_{\theta}$ and shape factor H. The definition of these parameters and their computer notation is

DELSTR
$$\delta^* = \int_0^\infty (1 - u/u_e) dy$$

THETA
$$\theta = \int_{0}^{\infty} u/u_{e}(1 - u/u_{e})dy$$

$$c_f = 2\tau_w/\rho u_o^2$$

RDELST
$$R_{\delta}^{*} = \delta^{*}u_{0}/v$$

RTHETA
$$R_{\theta} = \theta u_{0} / v$$

RZ
$$R_z = zu_0/v$$

In terms of transformed variables, δ^{\star} , θ and c_{f} can be written as

$$\delta^{\star} = \frac{z}{\sqrt{R_z}} \left[\eta_{\infty} - f_{\infty}/f_{\infty}' \right]$$

$$\theta = \frac{z}{\sqrt{R_x}} \int_0^{\eta_{ef}} \frac{f'}{f'_e} (1 - \frac{f'}{f'_e}) d\eta$$

$$c_f = 2 \frac{f''_w}{\sqrt{R_z}}$$

```
COMMON/BLOO/ NXT.NZT.NX.NZ,NP.NTR.ITMAX.IBDY.NPT.IZIG.ITUPB.ETAF.
                   VGP.A(101), ETA(101), PFTA(101)
      COMMON/REC1/ X(101).Z(81).UC(81).FZ(81).P1(81).P2(81).P3(101.81).
                   UE (101.81)
     1
      COMMON/BLCP/ DFLV(101) +F (101.81.2) +U(101.81.2) .V(101.81.2).
                   8(101,81,2)
      COMMON/PRNT/ IPRNT
      COMMON/INTP/ N.IEX(81)
      COMMON/SAVE/ ITIME.NXTD.NZTD.XC(101).ZC(81).CF5(81).HC(81).RTHETA
      COMMON/PLCS/ FC(101).UC(101).VC(101).BC(101)
      DIMENSION
                   FF(101.3).UU(101.3).VV(101.3).BB(101.3)
      DIMENSION PTHETA (81)
      P 585 (5, 101) NXPRNT
           = 0
      I^{TM}\Delta X = 10
      ITIME = 0
   5 4X
             = 1
      N7
             = 1
      N
            = 0
      TTIME = ITIME+1
      IFC ITIME .CT. 1 ) GO TO 6
      CALL IVP
    + nn 7 J = 1. NPT
      F(J. 1. 2) = FF(J. 3)
      U(J. 1.2)=UU(J.3)
      V(J. 1.2)=VV(J.3)
      R(J, 1, 2) = PR(J, 3)
     LUNT INFOE
   10 IF ( IPENT .LT. 2 .AND. TTIME .GT. 2) WPITE(6.110) NX.N7.X(NX).
              7(47)
      IF ( ITUPA .FD. 0 ) GO TO 12
      IF ( NY .FQ. 1 .AND. NZ .EQ. 1 ) 00 TO 90
   12 CONTINUE
      IF ( N7 .FO. 1 .AND. NX .GT. 1 ) GC TO 90
      •
           = 0
      *GR - 0
   20 17
           = 1+1
      IFIT .NE. ITMAX) GO TO 30
      WP TT = ( 6. 100 ) N7
      ספ מד ספ
   30 IF(MY .GF. NTR) CALL SPOY
      TE ( MX .GT. 1 ) GO TO 50
      CALL TONIT
      so to co
   50 CALL COFFG(IT)
   AD CALL SCLV3(IT)
      CILL SMOOTH
  CHECK FOR CONVERGENCE
      TEINT .CF. MTR) GO TO 70
C-- I AV TNAF FI OW
      TELEPSICELVILLE .GT. C. 00011GC TO 20
      SE TO PO
C--THERHLENT FLOW
  70 TELARSITELVILLE .LT. 2.55-04 .CF. ARSIDELVILLE
```

```
1 (V(1,M7,2)+0.5*DFLV(1))) .LT. 0.02) GO TO 80
    GO TO 20
 80 IFIND .ED. NPT) GO TO CO
    IF(ABS(U(MP-2.N7.2)/U(NP.N7.2)-1.0) .LT. 0.0015
   1 .AND. APS(V(MP.NZ.2)) .LF. 0.001) GC TO 90
    IFIIGROW.GF.2) GOTO FC
    IGPOW = IGPOW+1
    WR [ T = (6. 120)
         = 2
    1.1
    CALL GROWTH(LL)
    מה בש שנו
SO TALL CUTPUT
    IF ( NX .GF. NXPRNT ) IPRNT = 0
    TF ( ITIME .ST. 2 ) GO TO 10
    IF ( ITIME .ED. 2 .AND. NZ .EQ. 3 ) GC TO 96
    IF ( ITIME .NF. 1 ) GO TO 10
    IF! M .LT. 3) GO TO 92
    no ci J=1.NPT
    FF(J, 3)=F(J, 3, 2)
    1111(J.3)=U(J.3.2)
    VV(J.3)=V(J.3.2)
    RR(J,3)=R(J,3,2)
    CONT INUF
91
    CO TO 5
    M = M + 1
c 2
    90 03 J=1.MPT
    FF(J,M) = F(J,M,2)
    UU(J.M) = U(J.M.2)
    VV(J,M) = V(J,M,2)
    RR(J,M) = R\{J,M,2\}
93
    CONTINUE
    IF ( M .LT. 3 ) GO TO 10
    NN Z
        = 2
    CALL REPROF ( MPT. FF. UL. VV. RB. NNZ )
    41.7
          = 3
    GO TO 10
        = 1
    M
96
    on 97 J = 1.NPT
    EE(J.M) = C(J)
    UU(J,M) = UC(J)
    VV(J.M) = V(IJ)
    PRIJ.M) = P(IJ)
97
    PP 98 K = 1.2
         = M+1
    00 98 J = 1.NPT
    EF(J,M) = F(J,K,2)
    IJ(I\{J,M\}) = U\{J,K,2\}
    VV(J.M) = V(J.K.2)
   RP(J,M) = R(J,K,2)
    M7 = 1
    TALL DEPONE ( NOT, FF. UU, VV. AB, NZ )
    ITIME = TTIME+1
    IF ( IPPNT .LT. 2 .AND. ITIME .GT. 1) WRITE(6.110) NX.NZ.X(NX).
            ZINZI
    CALL CHITPUT
```

100 FORMAT(1H0.16X.32HITERATIONS EXCREDED ITMAX AT NZ=.13)
101 FORMAT(15)
110 FORMAT (1H0.4HNX =.13.5X.4HNZ =.13.5X.3HX =.F10.5.5X.3HZ =.F10.5)
120 FORMAT(1H0.2X.*ROUNDARY LAYER HAS GROWN*)
121 FORMAT (1H0.5X.5HNX = .13.5X.8HX(NX) = .F10.5.5X.5HXI = .F10.5/
11H0.FX.3H J .5X.5HZ (J).5X.8HV (WALL).10X.2HCF.
2 13X.1HH.12X.6HRTHETA.13X.2HUE.8X.6HEXTRAP /)
122 FORMAT (1H .5X.13.2X.F11.6.5(2X.E13.6).5X.11)
FAND

```
SURFOUTINE COEFG(IT)
  COMMON/BLCC/ NXT.NZT.NX.NZ.NP.NTP.TTMAX.IRDY.NPT.IZIG.ITUFB.ET AE.
 1
                VGP . 4 (101) . E TA (101) . DE TA (101)
  COMMON/BLC1/ X(101),Z(81),UC(81),PZ(81),P1(81),P2(81),P3(101,R1),
                UE (101,81)
  COMMON/BLCC/ $1(101),$2(101),$3(101),$4(101),$5(101),$6(101),
                R1(101),R2(101),P3(101)
  CCMMON/PLCP/ DELV(101), F(101,81,2), U(101,81,2), V(101,81,2).
                B(101.81.2)
  COMMON/7GZG/ KALC(101)
  COMMON/SOVE/ NZIG+NZIGS
  U(NP+NZ+2) = UE(NX+NZ) / UO(NZ)
  UOB = 0.5*(UO(NZ)+UO(NZ-1))
  IF ( IT .GT. 1 .AND. NZIGS .EQ. 1 ) GC TO 6
  NZIG = 0
  KALC(1)=0
  77 4 J=2.NP
  UP = 0.5*(U(J,NZ,2)+U(J-1,NZ,2))
  IF(UB .GE. 0.0) GD TO 4
  N7IG = 1
  KALC(1)=1
  GO TO 6
  CONTINUE
  DELX
        = X(NX)-X(NX-1)
  DELZ
         = Z(N7)-Z(NZ-1)
  7 R
         = 0.5*(Z(NZ)+Z(NZ-1))
  CEL
         = Z9/DELZ
  BELM
         = 78/(DELX)/UOB
  CFL 2
        = 0.5*CFL
  CFL 4
        = 0.25*CEL
  PIH
         = 0.5*P1(NZ)
  P2H
         = 0.5 * P2(N7)
  PIR
         = 0.5*(P1(NZ)+P1(NZ-1))
  P2R
         = 0.5*(P2(NZ)+P2(NZ-1))
  PIRH
        = 0.5*P18
  P 38
         = 0.25*(P3(NX,NZ)+P3(NX-1,NZ)+P3(NX,NZ-1)+P3(NX-1,NZ-1))
  DUSDZ
          = 0.25*(U(NP,NZ,2)**2-U(NP,NZ-1,2)**2
            +U(NP,NZ,1)**2-U(NP,NZ-1,1)**2)/DELZ
  אַרער
          = 0.5*(U(NP,NZ,2)-U(NP,NZ,1)+
            U(NP.NZ-1.2)-U(NP.NZ-1.1))/DELX
 1
  PSR
            ZB*(DUSDZ+DUDX/UOB)+P2B*(U(NP,NZ,2)**2+U(NP,NZ,1)**2
            +U(NP,NZ-1,2)**2+U(NP,NZ-1,1)**2)*0.25
   IF(NZ .FQ. NZT) GO TO 10
  M7P1 = MZ+1
  F 1
         = Z(NZ)*( (Z(NZP1)-Z(NZ))/(Z(NZP1)-Z(NZ-1))/DELZ )
  51H
         = 0.5**1
  F 2
         = Z(NZ)/DELX/UD(NZ)
  F.3
        = Z(NZ)*DFLZ/(Z(NZP1)-Z(NZ-1))/(Z(NZ)-Z(NZP1))
  2387
         = 0.5*(P3(NX,NZ)+P3(NX-1,NZ))
  ענוטים
          = (U(NP,NZ,2)-U(NP,NZ,1))/DELX
  DUSDZ
          = 0.5*((U(NP+NZ+2)**2-U(NP+NZ-1+2)**2)*51
            +(U(NP,NZ,1)**2-U(NP,NZP1,1)**2)*E3)
  1
   D 387
          = Z(NZ)*(DUSPZ/Z(NZ)+DUDX/UC(NZ))+
            P2(NZ)*(U(NP,NZ,2)**2+U(NP,NZ,1)**2)*0.5
10 CONTINUE
   DO 50 J=2,NP
```

```
FR
         = 0.5*(F(J,NZ,2)+F(J-1,N7,2))
         = 0.5*(F(J,NZ,2)*V(J,NZ,2)+F(J-1,NZ,2)*V(J-1,NZ,2))
   EVB
  HR
         = 0.5*(U(J,NZ,?)+U(J-1,NZ,2))
         = 0.5*(U(J,N7,2)**2+U(J-1,N7,2)**2)
  1159
         = 0.5*(U(J.NZ-1.2)+U(J-1.NZ-1.2))
  (JP 2
  UB4
         = 0.5*(U(J.NZ.1)+U(J-1.NZ.1))
   VA
         = 0.5*(V(J,NZ,2)+V(J-1,NZ,2))
   DEDRV = (R(J,NZ,2)*V(J,NZ,2)-R(J-1,NZ,2)*V(J-1,NZ,2))/DETA(J-1)
         = 0.5*(F(J,NZ,1)+F(J-1,NZ,1))
  USP4 = 0.5*(U(J:N7,1)**2+U(J-1,N2,1)**2)
   TE(N7 .FR. NZT) GO TO 20
   IF ( N7IG .FQ. 1 ) GC TO 3C
        = F(J,N7,1)*V(J,N2,1)+F(J,N2-1,1)*V(J,N7-1,1)+
20 FVJ2
           (J,N7-1,2)*V(J,NZ-1,2)
         = F(J-1,NZ,1)*V(J-1,NZ,1)+F(J-1,NZ-1,1)*V(J-1,NZ-1,1)+
           F(J-1.NZ-1.2)*V(J-1.NZ-1.2)
         = 0.25 \div (F(J,N7-1,1)+F(J-1,NZ-1,1)+F(J,NZ-1,2)+F(J-1,NZ-1,2))
   FVP234= 0.5*(FVJ2+FVJ1)
        = 0.5*(U(J,NZ-1,2)**2+U(J-1,NZ-1,2)**2)
  11582
  11502
        = 0.5*(U(J,NZ-1,1)**2+U(J-1,NZ-1,1)**2)
  USPII = 0.5*(USB2+USB3)
        = 0.25 \times (U(J,NZ-1,1)+U(J-1,NZ-1,1)+U(J,NZ,1)+U(J-1,NZ,1))
  IJPK 1
  115J2 = U(J,NZ,1)**2+U(J,NZ-1,1)**2+U(J,NZ-1,2)**2
  U(J-1.NZ.1)**2+U(J-1.NZ-1.1)**2+U(J-1.NZ-1.2)**2
  USB234= 0.5*(USJ2+USJ1)
         = V(J-1.N7.1)+V(J-1.N7-1.1)+V(J-1.N7-1.2)
  VJ1
  V 12
         = V(J,MZ,1)+V(J,MZ-1,1)+V(J,MZ-1,2)
  VP224 = 0.5*(VJ2+VJ1)
         = R(J-1,N7.1) * V(J-1,N7,1)+B(J-1,N7-1,1) *V(J-1,N7-1.1)+
           P(J-1,N7-1,2)*V(J-1,N7-1,2)
   BVJ2
         = P(J, NZ, 1) * V(J, NZ, 1) + B(J, NZ-1, 1) * V(J, NZ-1, 1) +
           P(J,NZ-1,2)*V(J,NZ-1,2)
         = US94-2.0*US811
   C W 2
         = UB2-2.0*UBK1
   CM4
         = FR4-2.0*FR[]
   CMB
         = (RVJ2-RVJ1)/\Gamma FTA(J-1)
         = CM1*CFL-0.5*CFL*VB234*CM4+2.0*8FLM*CM2-CM8-P18*FVP234
   CNZJ
         + P 28+USP 234-4.0+P39
   CORRECTORENTS FOR THE REGULAR BOX.
   (1(J) = B(J, M7, 2)/DF^*A(J-1) + P1BH*F(J, N7, 2) + CEL4*(FB+CM4)
   52(J) = -P(J-1,NZ,2)/DFTA(J-1)+P1RH*F(J-1,NZ,2)+C5L4*(F8+CM4)
   $3(J) = PIBH#V(J.NZ.2)+C5L4#(VB+VB234)
   <4(J) = PIPH*V(J-1.NZ.2)+CEL4*(VB+VB234)</pre>
   $5(J) =-(P2P+CEL)*U(J.NZ.2)-BELM
   56(J) =-1028+0EL)+U(J-1.NZ.2)-RELM
   22(J) = CN3J-(DEPBV+P1E*FVB-(P2R+CTL)*USB+CTL2*(VB*FB+VB234*FB
               +CM4* VP)-BFLM*2.0* LP)
   K 41 C ( J ) = 0
   CO TO 40
30 FF 2
         = 0.5*(F(J,N7-1,2)+F(J-1,N7-1,2))
         = 0.5*(V(J.N7-1.2)+V(J-1.N7-1.2))
   FVA4
         = 0.5*(F(J.N7.1)*V(J.NZ.1)+F(J-1.NZ.1)*V(J-1.NZ.1))
   HR 6
         = 0.5 \times (II(J.NZP1.1) + U(J-1.NZP1.1))
   ch6
         = 0.5*(F(J.NZP1.1)+F(J-1.NZP1.1))
         = 0.25*(U(J.M7.1)+U(J-1.N7.1)+U(J.N7.1)+U(J.N7.1)+U(J-1.N7.1))
   11R46
```

```
VR46 = 0.25*(V(J,NZ,1)+V(J-1.NZ,1)+V(J,NZP1.1)+V(J-1,NZP1,1))
   DFBV4 = (B(J,N7,1)*V(J,NZ,1)-B(J-1,NZ,1)*V(J-1,NZ,1))/DFTA(J-1)
   ^{\circ}1
        = F1H*(VB2*FR2-UP2**2)+F3*((UB4-UP6)*UP46-VB46*(FB4-FB6))
         = DRRV4+P1(NZ)**VR4-P2(NZ)*USB4+2.0*P3BZ
   r 3
         = -0.2+2.0*0.1-2.0*5.2*UB4
   COFFFICIENTS FOR THE ZIG-ZAG BCX.
   S1(J) = B(J,NZ,2)/PE^{A}(J-1)+P1H*F(J,NZ,2)+F1H*(FB-FB2)
   S2(J) = -B(J-1,NZ,2)/DFTA(J-1)+P1H*F(J-1,NZ,2)+E1H*(FB-FB2)
   93(J) = P1H*V(J,NZ,2)+F1H*(VB+VP2)
   (4(J) = P1H*V(J-1,NZ,2)+E1H*(VB+VB2)
   SF(J) = -P2(NZ)*U(J,M7.2)-E2-F1*U(J.NZ.2)
   56(J) = -P2(N7)*U(J-1,NZ,2)-E2-F1*U(J-1,NZ,2)
   02(1) = 03-(DEFBV+P1(N7)*FVP-P2(NZ)*USB-2.0*E2*UB-E1*(UB**2
             - VR*F8-VR2*F8+F82*VB11
   KALC(J)=!
40 R1(J) = F(J-1,N7,2)-F(J,N7,2)+DFTA(J-1)*UP
   R3(J-1)=U(J-1,NZ,2)-U(J,NZ,2)+DFTA(J-1)*VB
50 CONTINUE
   IF ( IT .=Q. 1 ) NZIGS = NZIG
   R3(NP)= 0.0
   r1(1) = 0.0
   F2(1) = 0.0
   PETURN
   END
```

```
SUBFOUTING FORY
   COMMON/BLCO/ NXT.NZT.NX.NZ.NP.NTR.ITMAX.IBDY.NPT.IZIG.ITURB.ETAF.
                 VGP . 4 (101) . ETA (101) . DETA (101)
   COMMON/BLC1/ X(101), Z(81), UC(81), PZ(81), P1(81), P2(81), P3(101,81),
  1
                 UF (101.81)
   COMMON/BLCP/ DFLV(101), F(101,81,2), U(101,81,2), V(101.81,2),
                 P(101.81.2)
   COMMON/RECO/ RE-CF-RTH
   DIMENSION EDV(101)
   DIMENSION TB(101)
   GAMTR = 1.0
   TE ( ITURE .NF. 0 ) GO TO 12
   unt
         = 0.0
   UF 1
         = 1.0/U^{-}(NX,NTP-1)
   DC 10 I=NTR,NZ
   บาว
         = 1.0/UE(NX.I)
         = 1001+(1001+1002)*0.5*(Z(I)-Z(I-1))
   UOI
10 001
         = UP 2
   GG
         = 8.35F-04*UF(NX,NZ)**3 / (UF(NX,NTR-1)*Z(NTR-1)**1.34
   EYPT = GG*FL**0.66*(Z(NZ)-Z(NTP-1))*UCI
   'F ( EXPT .LF. 10.0 ) GAMTP = 1.0 - EXP(-EXPT)
12 \text{ CUM1} = 0.0
         = ((1.N7.2)/U(NP.N7.2)*(1.0-U(1.NZ.2)/U(NP.N7.2))
   Fl
   DO 13 J=2,MP
         = U(J,N7,2)/U(NP,NZ,2)*(1.0-U(J,N7,2)/U(NP,NZ,2))
   -2
   SUMI
         = S(M) + (F) + F(J) + A(J)
13 F1
         = = 2
   THETA = SORT(Z(NZ)/(RL*UD(NZ)))*SUM1
   r =
         = UF(NX.NZ) + THF TA+PL
   TE ( PT .LE. 425.0 ) GO TO 14
   TE ( FT .GT. 6000.0 ) GD TO 15
   XPT
         = 9^{-}/425.0-1.0
   DT
         = 0.55*(1.0-5XP(-0.243*SQPT(XPI)-2.98*XPI))
   40 TO 20
14 PI
         = 0.0
   GR TO 20
15 PT
         = 0.55
20 \text{ } \text{FLG} = 0
   P72
         = SOPT (HO(NZ)*Z(NZ)*PL)
   R74
         = SOPT (PZ2)
   VMAX
         = V(1,N2,2)
   nn 30 J = 2.NP
   IF(ABS(V(J.N7.2)).GT.ABS(VMAX)) VMAX= V(J.NZ.2)
30 CONTINUE
   ECVC = 0.0168*(1.55%(1.C+PT))*P Z2*(U(NP,NZ,2)*ETA(NP)-F(NP,NZ,2))
  1 * GAM TF
         = 1
80 Iffield . FO. 1) GO TO 50
   PPLUS = (P2(NZ)/RZ4)*(UF(NX.NZ)/UC(NZ))**2*(1.0/ABS(VMAX)**1.5)
         = R74*FTA(J)*SOPT(APS(VMAX)*(1.0-11.8*PPLUS))/26.0
         = 1.0
   IF(YOA .LT. 4.0) EL = (1.C-EXP(-YCA))**2
   EDVI = 0.16*972*ABS(V(J,NZ,2))*FL*GAMTR*ETA(J)**2
   IF(EDVI .LT. EDVO) OF TO 100
```

100 - T - 100 - 10

```
TELC = 1
90 FOV(J)= FOVO
    G2 TO 110
100 ENV(J)= ENVI
    IF(J.LF.2) GOTO 110
    IF(FDV(J).GT.FDV(J-1)) G0T0 110
    =nv(J+1)+(Env(J-1)+Env(J-2))*VGP
    IF(EDV(J).LT.EDVO) GOTO 110
    FOV(J) = FOVO
    IFLG = 1
110 8(J.NZ.2) = 1.0+EnV(J)
         = J+1
   IE(J .LF. NP) GO TO 80
   NDM1 = NP-1
   TR(1) = P(1.N7.2)
   nn 150 J = 2.NP
   TR(J) = (P(J-1,NZ,2)+P(J,NZ,2))*0.5
150 CONTINUE
    TR(NP) = TR(NPM1)
    nn 170 J= 2,NPM1
   B(J.N7.2) = (TB(J)+TB(J+1))*0.5
170 CONTINUE
    R(NP,NZ,2)=R(NPM1,NZ,2)
    Nanted
    END
```

```
SURPOUTINE GRID
COMMON/BLCO/ NXT.NZT.NX.NZ.NP.NTR.ITMAX.IBDY.NPT.IZIG.ITURB.FTAE.
VGP.A(101).ETA(101).DETA(101)
```

```
SURPOUTINE GROWTH(LL)
    COMMON/PLOD/ NXT.NZT.NX.NZ.NP.NTR.ITMAX.IBDY.NPT.IZIG, ITUFB, ET AF.
                  VGP.A(101).ETA(101).DETA(101)
   1
    COMMON/BLCP/ DELV(101) +F (101 + £1 + 2) +U(101 + 81 + 2) +V(101 + 81 + 2) +
                  8(101.81.2)
    NPT
          = NP
    ND 1
         = NP+1
    NPM\Delta X = 101
    IF ( LL .=0. 1 ) GO TO 95
         = NP + 2
    MP
    IF ( NP .GT. NPT ) NP=MPT
    MPMAX = MP
0500100 J = MP1.NPM4X
    =(J,NZ,2) = U(NPC,NZ,2) + (5TA(J)-ETA(NPO))+F(NPC,NZ,2)
    11(J.NZ.2) = U(NPO.NZ.2)
    V(1.N2.2) = 0.0
    B(J,M7,2) = P(NPO,N7,2)
100 CONTINUE
    RETURN
```

CNE

```
SUBPOUTINE ICONZ
   COMMON/BLCO/ NXT,NZT.NX.NZ.NP.NTR.ITMAX.IBDY.NPT.IZIG.ITURB.FT AE.
                VGP - 4 (101) - ETA (101) - DETA (101)
   COMMON/RLC1/ X(101),Z(81),UO(81),RZ(81),P1(81),P2(81),P3(101,81),
                UE (101.81)
  1
   COMMON/RLCP/ DELV(101), F(101,81,2), U(101,81,2), V(101,81,2),
                B(101,81,2)
   CCMMON/RLCC/ $1(101).$2(101).$3(101).$4(101).$5(101).$6(101).
                R1(101),R2(101),R3(101)
  1
   RFL = 0.0
   IF(NZ .GT. 1) BEL = 0.5*(Z(NZ)+Z(NZ-1))/(Z(NZ)-Z(NZ-1))
         = Pl(NZ)+BEL
   D 20
         = P2(NZ)+BEL
   00 3C J=2.NP
DEFINITION OF AVERAGED QUANTITIES
   ER
         = 0.5*(F(J,NZ,2)+F(J-1,NZ,2))
   UB
         = 0.5*(U(J,NZ,2)+U(J-1,NZ,2))
   VP.
         = 0.5*{V(J,N7,2)+V(J-1,N7,2)}
         = 0.5*(F(J,NZ,2)*V(J,NZ,2)+F(J-1,NZ,2)*V(J-1,NZ,2))
   FVP
   HER
         = 0.5*(U(J.N2.2)**2+U(J-1.N2.2)**2)
   DF^{Q}BV = (R(J,NZ,2)*V(J,NZ,2)-B(J-1,NZ,2)*V(J-1,NZ,2))/DETA(J-1)
   IF(N7 .GT. 1) GO TO 10
   CFB
         = 0.0
         = 0.0
   CUB
   CAB
         = 0.0
   CRR
         = -P2(NZ)
   GC TO 20
         = 0.5*(F(J,NZ-1,2)+F(J-1,NZ-1,2))
10 CFB
   CUR
         = 0.5*(U(J.N?-1.2)+U(J-1.N?-1.2))
   CVR
         = 0.5*(V(J,NZ-1,2)+V(J-1,NZ-1,2))
   CEVB
         = 0.5*(F(J,N7-1,2)*V(J,N7-1,2)+F(J-1,N7-1,2)*V(J-1,N7-1,2))
         = 0.5*(U(J,NZ-1,2)**2+U(J-1,NZ-1,2)**2)
   CHISE
   CDERBV= (B(J,NZ-1,2)*V(J,NZ-1,2)-B(J-1,NZ-1,2)*V(J-1,NZ-1,2))/
           DETA(J-1)
   CLB
         = CDFRBV+P1(NZ-1)*CFVP-P2(NZ-1)*CUSB+P3(NX,NZ-1)
         = -P3(NX.NZ)+BFL*(CFVB+CUSB)+CLB
   CRB
COFFFICIENTS OF THE DIFFERENCED MOMENTUM FOUATION
20 (J_1) = B(J_1N7, 2)/PETA(J_1) + (P1P*F(J_1N2, 2)-BEL*CFB)*0.5
   S?(J) = -B(J-1.NZ.2)/DFTA(J-1)+(P1P*F(J-1.NZ.2)-BEL*CFB)*0.5
   S3(J) = 0.5*(P1P*V(J,NZ,2)+BEL*CVB)
   (4(J) = 0.5*(P1P*V(J-1,N7,2)+BEL*CVE)
   SS(J) = -P2P*U(J,NZ,2)
   S6(J) = -P2P*U(J-1,NZ,2)
DEFINITIONS OF RJ
   P1(J) = F(J-1,NZ,2)-F(J,NZ,2)+DETA(J-1)*UB
   =3(J-1)=U(J-1.NZ.2)-U(J.NZ.2)+DETA(J-1)*VB
   P2(J) = CRB-(DERBV+P1P*FVB-P2P*USE-BFL*(CFE*VE-CVB*FB))
30 CONTINUE
   91(1) = 0.0
   P2(1) = 0.0
   P3(NP)= 0.0
   PETURN
   END
```

```
CUREDUTINE INPUT
     COMMONIAL COLINXTINZTINXINZINFINTRITMAXIEROVINPTIZEGIETUERI CTAFI
                  VCP.A(101) ATTA(101) AFTA(101)
     COMMONIZALCI/ X(101),7(21),UP(81),PZ(81),PI(81),PZ(91),P3(101.81),
                  UF (101,81)
     COMMON/RLCP/ DFLV(101), F (101.81.2), U(101.81.2), V(101.81.2),
                  3(101,81,2)
     COMMONIABLODI RE .CF .RTH
     COMMON/PRNT/ IPRNT
     COMMON/SAVE/ ITIME, NXTO, NZTO, XC(101), ZO(81)
     DIMENSION TITLE (20)
     PI = 3.141593
     MPT = 101
     ITMAX = 10
     TAE = 8.0
     750
           = 2
     INTE
           = 2
     PL
           = 1.0
     RΔ
           = 2.4/(33.0*3.45)
     1=11TIME .50. 1)GO TO 60
     NX^{+}=NX^{+}D
     M7 T=N7 TO
     DO 40 I=1.NXT
 40
     X(I)=XU(I)
     DP 42 I=1.N7T
 42
     7(1)=20(1)
     GD TO 80
     TTUPR = 0
 60
     READ (5. 270) TITLE
     SEAD (5. 260) NXT.NZT.NTP.IRCY.RL.IPRNT.DETA(1).VGP.CF.RTH
     i \in (NTR.EQ.1) [TUPR = 1
     9840 (5, 290) (X(I), I=I,NXT)
     FFAD (5. 290) (Z(I), I=1.NZT)
     NXTO=NXT
     MATCHNIT
     00 70 !=1.NXT
    XU(1)=X(1)
 70
     no 72 I=1.477
 72
    70(1)=7(1)
     71 = 7(1)
     nn 74 I=1.3
 74
    7(7) = 71-9.01*(3-1)
     NXT=1
     N7 T = ?
     00 100 !=1.NXT
     X([]) = X([]) * 23.0
 100 CONTINUE
     IF( ITIME .EQ. 1) GO TO 120
     WRITE(6. 330) TITLE
     WRITE(6. 340) MXT.NZT.NTR.IPDY
     WPITEL6. 3421 PL. DETA(1), VGP. CF. ETH
     PC 140 I = 1.NZT
120
     ## (1. I)=1.0
     P3(1.11=0.0
     PRESSURE GRADIENT PARAMETER FOR STEADY STATE
```

```
P2(1) = 0.0
     P1(!) = 0.5*(1.0+P2(!))
 140 00(1) = 1.0
     TE( !TIME .EQ. 1) RETURN
     SAMPLE TEST CASE
     on 150 K = 1.4XT
     no 150 I = 1.87T
     FUNC = 1.0
     n= 0.0
     FUNT = 1.0
          = C.C
     TE(2(1) .GF. 1.34) GO TO 143
     FINIC = FIN(PI/2.0*((Z(I)-1.24)/0.1))
     D^{c} = PI/2.0/0.1*CDS(PI/2.0*((Z(I)-1.24)/0.1))
143 IF(X(K) .GT. 1.98) GO TO 145
     EIJN'T = SIN(PI/2.0*(X(K)/1.98))
          = P1/1.98/2.0*CCS(P1/2.0*(X(K)/1.98))
145 \text{ UF(K.I)} = 1.0-RA*X(K)*(7(I)-1.24)*FUNC*FUNT
     DUFDX = (-PA*X(K)*FUNC-BA*X(K)*(Z(I)-1.24)*DF)*FUNT
     DUFDT = {-BA*(Z(I)-1.24)*FUNC)*FUNT+DT*(-BA*(Z(I)-1.24)*X(K)*FUNC)
     P3(K, I)=7(I)/U0(I)**2*(UE(K, I)*PUEDX+DUEDT)
150 CONTINUE
    WPITE (6. 322) NXT
    WRITE (6. 326) ( X(I),I=1,NXT )
    WRITE (6, 324) NZT
    WRITE (6. 326) ( Z(I),I=1,NZT )
    RETURN
260 FORMAT( 615.5F10.0 )
 270 FORMAT(20A4)
290 FORMAT(8F10.0)
322 FORMAT (///1HO, 27HTAPLE OF INPLT X FROM 1 TO , 13 / )
324 FORMAT (1HO.27HTARLE OF INPUT 7 FROM 1 TO , I3 /)
326 FORMAT (1H . 3X, 12F10.5 )
330 FORMAT(1H0, 2044)
340 FORMAT( ///1H0,12H** CASE DATA/1H0,3X,6HNXT =,13,14X,6HNZT =,13,
             14X,6HNTR =,13/ 4X,6H1BDY =,13
   1
                                                 +14X+6HIZIG = +I3+14X+
             6HITURB=,13)
342 FORMAT( 1H .3X,6HRL
                           =,E14.6,3x,6HDETA1=,F14.6,3x,6HVGP =,E14.6/
             4X.6HCF =.E14.6.3X.6HRTHET=.E14.6 )
 350 FORMAT(1H0.3X.6HBB
                         = .F14.6.3 \times .6HBA = .F14.6.3 \times .6HBL
                                                              = , 514.6.
   1
             3X, 6HZO
                      =,F14.6/)
     END
```

```
CHECOUTTME TUP
   COMMONIZAL COIZ NIXT, NIZT, NIX. NIZ, NIE, NITR, TITMAX, TROY, NPT, TZ 15, TTHER, ET &F.
                 VGP.4(101), FTA(101), PFTA(101)
  1
   COMMON/FLC1/ X(101).7(F1).LC(81).F7(81).P1(81).P2(81).F3(101.81).
                 IF (101. P1)
   COMMON/ALCP/ DELV(101), F(101,81,2), U(101,81,2), V(101,81,2).
                 91101.81.21
  COMMONIVATION REPLETT
   COMMONITED ITIME, NATE, NATE, XC(101), ZC(81)
  rimenciam UU(101). FF(101). GG(101). SHEAR(101)
   DATA PIRKIC/3.14159265.C.41.2.0/
   TE ! ITHER .FO. 1 ) GO TO B
  LAMINAR DENFILS
  CALL GEID
        = 1
   TTAMPO = 0.25 = TA (NP)
   FTAULE 1.5/FTA(NP)
  70 3 J=1.NP
  STAR = FTA(J)/FTA(NP)
   FTAE? = FTAR**2
  F(J.N7,2) = FT4MPQ#FTAP2# (3.0-0.F#FT4P2)
  11(3.N7.2) = 0.5*FTAP*(3.0-FTAP2)
  V(J.N7.2) = FTAU15+(1.0-FTAR2)
  P(J,NZ,2) = 1.0
3 CONTINUE
  RETURN
   THREULENT DROFTLE
 P R7(N7)= UF(NX,N7)*7(N7)*PL
   COPZ = CORT(PZ(NZ))
   CFD2 = 0.5 * CF
   SOCED 2= SORT(CED 2)
  HTMI = SOCED2
   SRXCF2= SQF7*SOCF02
   SCEDOK = SOCEDS/OK
         = 0.1
   YY
         = SOCFOS/PK
   A NI
   A &
         = \Delta N * \Delta M
  CCI
         = -1.9123016*AA+11./12.*AN
   rr 2
         = AN-3.05603#AA
   003
         = -1.5*44
   CC4
         = 1.0/AN-C-ALOG(SQCFC2*RTHFT1)
   1 4 1
         = CC1+0.5*CC2*CC4+C.25*CC3*CC4**2
   142
         = 0.5*(502+663*064)
   443
         = 0.25*003
10 YLOG = \DeltaLOG(YY)
   EEC
         = YY+(AA1+AA2*YLCG+AA3*YLCG**2)
   UE
         = 1.0-(4A2+2.C*AA3*YLCG)/YY
   777
         = FFC /DF
   Y Y
         = YY - DYY
   TE(ABS(DYY) .LT. 0.00001) GD TD 20
         = 1^{+}1
```

```
TF( ] T. LT. 10) GO TO 10
30 CONTINUE
         = 0.5+(ff 4+4LfG(YY))
    DUEL TAR E THE TIVY
          = POEL TA / SOR 7
    IT ( IT IME .GT. 1) WRITE (6.150) PIE .PDFLTA .ETAF ,CF . RTHFT1
    DEGPRE FO. 0/SP XCF2
    CALL GE ID
    00 30 J=1.NP
    TELETALLY .GT. PEGAPK) GO TO 40
    "10 = G1 = J
    50 TO 50
40 MPFC1 = J-1
   415 = C.2 = J
THE THE FR. EDD DRAFTLES
          = = TA (NOEGI) /FTA (ND)
50 77
          = FTG (NREG 11+ SP XCE 2
    22
          = 1./FK+(ALOG(R2)+C+P1E+().-CCS(P1+ZZ))+ZZ+ZZ+(1.-ZZ))
    3 1
    ~
          = 0.09
          = 0
AT FEC
          = P 1+( Y-5 TAN( P 2+( Y)
          = Q 1-0 2/(1.+(P 2*(Y)**2)
          = FE( /DE /( Y
    774
          = (Y*([.- 7(Y)
    ~ V
    TELTACION Y). LT. 0.001 . CF. 17.67.15) GCTO 70
          = 1 7+1
    22.42
70 ITSTH = CV# CD XCE?
    y = v
         = CULEUS# CD XCES
    TO BE JEI. NOTEL
    1(1, 17, 1) = 500502+(6764(LTATM+FTA(J))/CY)
90 (11.17.2) = VTM/(1.0+(NTNTM*FTA(J))**2)
    TOTAL TO HOLDE TO 100
          = FTE ( YOUGH ) /ETA (NIP)
    2.7
          = "IINPEGI,NI.2) / SC FOZK- (ALCC (SR XC EZ #ET A (NR EGI))+DIE#
           (1.-005101#77))+72*22*11.-22))
          = DIC+DI/FTA (ND)
    VI
    שנ פני א=+ שבני איים
    77
          = ETA ( ) ) / FTA ( P.D )
    -- CANCE COC(PT#77)
    TINAMO = SOFTEL .- COSANG**2)
    90 V(J,N7,7) = SCSO2K*(1./STA(J)+VJ*STNANG+ZZ/STA(NP)*(2.-3.*771)
 THE THE TE PROFILE
100 5054
         = 2.*CV** 2*50 XCF2
    F11.47.21 = 0.0
    DO IIO J#2.NPFG1
110 - 5( ). 47. 21 = 50 5 50 24 (F T& ( J) 75 44 TAK ( TKTM#FTA ( J) ) -1.07 FDFN#
            ALOG(1.0+(UTM.TM+ETA(J))++21)
    TELLABELT .FO. NPT OF TO 130
    TTA1 = FTA(NFFG1)
    c 1
           # DIF#FTA (NP) /FT
          = FTE 1/574 (ND)
    7 7
    FC = F( ) F F G 1. N 7 . 2) - SC F O 2 K + (F T A ) + (A ) C G (S F X C F 2 + E T A ) - 1 . O + C + P I F ) - F J *
              <!!!(>1+77)+F TA1+22+27+(1./3.-27/4.))
```

150 FORMAT(1H0, 4HPTF=,E14.6.3x,7HRPFLTA=,E14.6,3x,5HETAE=,E14.6,3x, 1 3HCF=,E14.6.3x,8HRTHFTA1=,E14.6)

```
TURTUO PINITUORRUZ
    COMMON/PLCC/ NXT,NZT,NX,NZ,NP,NTR,ITMAX,IBDY,NPT,IZIG,ITUFS,ET AE.
                  VGP, A(101), ETA(101), PETA(101)
   1
    COMMON/RLC1/ X(101),Z(81),UC(81),RZ(81),P1(81),P2(81),P3(101.81),
                  UF (101,81)
    TOMMON/FLCP/ DELV(101), F(101.81.2), U(101.81,2), V(101.81.2),
                  R(101, P1.2)
   1
    COMMON/ZGZG/ KALC(ICI)
    COMMON/PLCD/ RL.CF.RTH
    CCMMON/PPNIT/ IPRNT
    COMMON/INTP/ N. IFX(81)
    COMMON/SAVE/ ITIME.NXTO.NZTO.XC(101).ZC(81).CFS(81).HS(81).FTHETA
    DIMENSION RETHETA (RI) . PTHT(81) . NPK(81)
    TIMENSION FE(101.2). UU(1C1.2). VV(101.2). BB(101.2)
    PIXENSION =N(101), UN(101), VN(101), 8N(101)
        = 177-2
    A17 7
    TEXTOD= 0
    TEXINZ) = C
    1= ( | T | MF .LT. 3) GC TO 5
    WPITE(6. 220 )
    MPMI = MP-1
         = 1
   48 TT = (6. 230 ) J. = TA (J), F (J, NZ, 2), U(J, N7, 2), V(J, NZ, 2), B(J, N7, 2),
                     KALC (J)
   MDMX = ND-7
   WFITF(6. 230 )(J.ETA(J),F(J.NZ.2),U(J.NZ.2),V(J.NZ.2),P(J.NZ.2).
                    KALC(J), J=NPM3, NP)
  E COVIT IN DE
    MPK(NZ)=ND
    IF ( MZ .EQ. 1 .AMD. NTD .NE. 1 ) GC TO 90 C1 = SOPT( Z(NZ) / (RL*UF(NZ)) )
    DFL STF = [ ]* ( ETA (NP ) - F (NP , NZ , 2) / U (NP , NZ , 2) )
    C F
         =2.0*V(1.NZ,2)/(SORT(FL*UC(NZ)*Z(NZ))*(UE(NX,NZ)/UC(NZ))**?)
    CES(M7) = 55
    ROFL ST= UF(NX,NZ)*DELSTR*RL
    SIIM 1 = 0.0
          = U(1.NZ,2)/U(NP,NZ,2)*(1.0+U(1,NZ,2)/U(NP,NZ,2))
    -1
    DO 10 J=2,NP
    E2
          = U(J.NZ.2)/U(NP.NZ.2)*(1.0-U(J.NZ.2)/U(NP.NZ.2))
    SUM1 = SUM1+(F1+F2)*A(J)
 10 =1
           = 52
    THETO = (1#5UM1
    PITHETA(NZ) = UT(NX.NZ)*THETA*RL
    RTH = PTHETE(NZ)
          = DEL STR /THE TA
    HS(NZ) = H
    TE ( IFXTPP .GT. O .AND. NZ .FC. NZT ) GO TO 190
90 CALL GPOWTH(1)
    IF (ITIME .LT. 3) GO TO 100
    IF ( IPRNT .GT. 1 ) GO TO 100
    WRITE ( 6. 242 ) DELSTR. THETA, CF. PRELST, RTHETA (NZ), H.
                       UF (NX,NZ)
100 IF ( NZ .FQ. NZT ) GO TO 190 IF(ITIME .LT. 3) GO TO 140
    TEC N7 .LT. NZ2) GO TO 140
```

```
= N + 1
      on 120 J = 1.NPT
      FF(J,N) = F(J,NZ,2)/UF(NX,NZ)
      UU(J,N) = U(J,NZ,2)/UF(NX,NZ)
      VV(J,N) = V(J,NZ,2)/U^{2}(NX,NZ)
  120 \text{ BP(J,N)} = P(J,NZ,2)
      TE ( NZ .LT. (MZT-1)) GO TO 140
      nc 124 J = 1.NPT
      TF (U(J.M7.2) .LT. 0. ) GO TF 130
  124 CONTINUE
      GO TO 140
  130 \text{ P71} = 7(\text{N7T-1}) - 7(\text{NZT-2})
      DZ 2 = Z(NZT) - Z(NZT - 2)
      70 134 J = 1,NPT
      DF = FF(J, 2) - FF(J, 1)
      rac{1}{2} = UU(J.2) - UU(J.1)
      \nabla V = VV(J,2)-VV(J,1)
      PR = PR(J.2)-PR(J.1)
      FN(J) = FF(J,1)+DZ2*DF/DZ1
      UNIJ) = UU(J.1)+022*0U/071
      V^{N}(J) = VV(J,1) + DZ2 * DV/DZ1
      PN(J) = RR(J,1) + DZ2*DR/DZ1
  134 CONTINUE
      TEXTEP = 1
      !FX( \\Z+1) =1
  140 M7 = MZ+1
      IFTITXTRP .LE. 0) GO TO 148
      DD 142 J = 1.NPT
      KALC(J) = 2
      F(J,NZ,2) = FN(J)*UF(NX,NZ)
      H(J\cdot NZ\cdot 2) = UN(J)*UE(NX\cdot NZ)
      V(J,NZ,2) = VN(J)*UF(NX,NZ)
  142 \text{ P(J,N7,2)} = \text{PN(J)}
      TE ( TPENT .NE. 0) GO TO 5
      WRITE(6, 220 )
      MPM1 = NP-1
      WRITTELE. 230 )(J.FTA(J), F(J.NZ.2), U(J.NZ.2), V(J.NZ.2), R(J.NZ.2),
                       KALC(J).J=1.NPM1.3)
      J = MP
      WF TTE(6, 230) J.FTA(J),F(J,NZ,2),U(J,NZ,2),V(J,NZ,2),R(J,NZ,2),
                       KALC (J)
      GD TO 5
  148 IF (NX .GT. 1) GO TO 160
   INITIAL GUESS FOR NEXT STATION
      00 150 J=1.NPT
      F(J,NZ,2) = F(J,NZ-1,2)
      U(J.NZ.2) = U(J.NZ-1.2)
      V(J,NZ,2) = V(J,NZ-1,2)
  150 B(J,N7.2) = B(J,NZ-1.2)
       IF ( NX .FQ. 1 ) RETURN
C
    DETERMINE MP EOR 716746 SCHEME
       IF (N7 .FQ. NZT) GO TO 170
       TE (NX .EQ. 1) GO TO 170
       IF (NP .LT. NOK(NZ+1)) MP=NPK(NZ+1)
      PETURN
```

^

```
160 NP
          = NPK(NZ)
     IF (N7 .FQ. I) RETURN
     IF INP .LT. NPK(NZ-1)) NP=NPK(NZ-1)
     15 (1716 .EQ. 0) GO TO 17C
     IF IND .LT. MPK(NZ+1)) NP=NPK(NZ+1)
        = UF(NX,NZ)/UF(NX,NZ-1)
 170 UR
     DD 180 J = 1.NPT
     F(J,NZ,2) = F(J,NZ-1,2) * UR
     U(J,NZ,2) = U(J,NZ-1.2)*UR
     V(J,NZ,2) = V(J,NZ-1,2)*UR
     B(J.NZ.2) = B(J.NZ-1.2)
 180 CONTINUE
     DETURN
    IFINX .EO. NXT .AND. ITIME .FO. 1) RETURN
190
     IF ( IFFNT .NG. 2 ) GD TC 194
    XI = X(NX)/37.0
     WPITE ( 6. 250 ) NX. X(NX).XI
     nn 193 K = 1. NZ
     WPITE ( 6, 260 ) K, Z(K), V(1,K,2),CFS(K),HS(K),RTHFT&(K),
                      UF (NX.K) . IF X(K)
    1
 145 COMILIMIE
 194 TEINX .FQ. NXT .AND. ITIME .FQ. 3) STCP
           = NX+1
           = 0
     N
     Ŋ7
           = 1
     SHIFT.
     nn 210 K=1,NZT
     DC 200 J=1.NPT
     F(J,K,1) = F(J,K,2)
     U(J,K,1) = U(J,K,2)
     V(J,K,1) = V(J,K,2)
 200 B(J.K.1)= B(J.K.2)
 310 CONTINUE
     G0 T0 1.60
 . . . . . . .
 220 FIRMAT(1H0.2X.1HJ.4X.3HFTA.10X.1HF.13X.1HU.12X.1HV.13X.1HF.8X.
            4HKALC)
 230 FORMAT(1H . 13.F10.5,4914.6.16)
 242 FORMAT(1H0.7MDELSTR=.E14.6.3x.7H7HFT& =.E14.6.3X.7HCF
                                                                = . E14.6/
            1H . 7HR DEL ST=, F14.6.3X, 7HR THETA= . F14.6
                      =,F14.6,3X,7HUE
            1H . 7HH
                                           = .F14.6 / )
          110.5x.5HNX = .13.5x.8HX(NX) = .F10.5.5X.5HXI = .F10.5/
 250 =
    11H0, = 3H J ,5x,5HZ (J),5x,8HV (WALL),10x,2HCF,
       13X.1HH.12X.6HRTHFTA.13X.2HLE.8X.6HEXTRAP / 1
    FORMAT (1H +5X+13+2X+F11+6+5(2X+E13+6)+5X+11)
     EA D
```

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SHAROUTINE
                CMUU TH
    COMMON/PLCO/ NXT.NZT.NX.NZ.NP.NTR.ITMAX.IBDY.NPT.IZIG, ITURB.FT AE.
                  VGP , & (101) , F TA (101) , DE TA (101)
    COMMON/PLC1/ X(101),Z(81),UC(81),PZ(81),PI(81),PZ(81),P3(101.81),
                  UE (101.81)
    COMMON/PLCP/ DFLV(101),F(101,81,2),U(101,81,2),V(101,81,2),
   1
                  9(101.81,2)
    DIMENSION FS(101).US(101).VS(101).BS(101)
           = NP-1
    NOWI
           = MP-2
    NPM2
           = 1
    JMAX
    MWAX
           = V(1,NZ,2)
    90 \cdot 10 \text{ J} = 2.00
    15 ( V(J.NZ.2) .LT. VMAX ) GE TO 10
    VMAX
           = V(J,N7,2)
    JMAX
           = J
    CONT INUF
10
    TUUT
           = U(JMAX+1+NZ+2)-U(JMAX+NZ+2)
    rvJ1
           = V(JM!X+1,N7.2)-V(JMXX,N7.2)
    .15
           = JM\Lambda X + 2
    00 5C 1=72.4Nb
    JJ
           = J
    DUJ 2
           = U(J,NZ,2)-U(J-1,NZ,2)
    CVJ2
           = V(J, MZ, 2) - V(J-1, NZ, 2)
    UJPEOD = UJ2*UJ1
    Albedu = Al5*Al1
    IF ( UJPROD .LT. 0.0 .CP. VJPPCD .LT. 0.0 ) GC TC 30
    DUJI
           = DUJ2
    PVJ 1
           = DVJ2
20
    CONT INHE
30
   IF ( JJ .FO. NP ) RETURN
    P^{-} 4C J = JJ.NP
    FC(J) = 0.5*(F(J-1,N7,2)+F(J,N7,2))
          = 0.5*(U(J-1,MZ,2)+U(J,MZ,2))
    (1)2(1)
    (L)2V
          = 0.5*(V(J-1,NZ,2)+V(J,NZ,2))
    P'(J) = 0.5*(P(J-1,NZ,2)+B(J,NZ,2))
40
   CONT INUE
    TO SC J= JJ.NPM1
    F(J, N7, 2) = 0.5*(FS(J)+FS(J+1))
    U(J.NZ.2)=0.5*(US(J)+US(J+1))
    V(J,NZ,2)=0.5*(VS(J)+VS(J+1))
    B(J,NZ,2) =0.5*(BS(J)+PS(J+1))
50
    CONTINUE
    V4'9
           = -V(NP-1,N7,2)+(U(NP,N2,2)-U(NPM1,N7,2))/4(NP)
    IF(ABSIVNP) .LT. APSIV(NP.NZ.2))) V(NP.NZ.2) = VNP
    DETURN
    ENID
```

```
SUBFOUTINE REPROF ( NPT. FF. UU. VV. BB. N )
   COMMON/PLC1/ X(101).Z(81).UD(81).PZ(81).PI(81).P2(81).P3(101.81).
                 UE(101.81)
   COMMONIPLED/ OFLV(101), F(101,81,2), U(101,81,2), V(101,81,2),
                 3(101,81,2)
   COMMON/PRNT/ IDRNT
    COMMON/BLCS/ FC(101).UC(101).VC(101).BC(101)
                 FF(101,3),UU(101,3),VV(101,3),RB(101,3)
    DIMENSION
    DIMENSION
                 FA (101,2), UA (101,2), VA (101,2), BA (101,2)
    DIMENSION
                 FN(101).UN(1C1).VN(101).BN(101).DX(2)
         = N+1
    DD 100 I = 1.2
    no ico J = I,MPT
    F(J,I) = 0.5*(FF(J,I)+FF(J,I+I))
   04(J,I) = C.5*(UU(J,I)+UU(J,I+1))
   VL(J,I) = 0.5*(VV(J,I)+VV(J,I+1))
    BA(J,I) = 0.5*(PP(J,I)+PP(J,I+1))
100 CONTINUE
    no 140 J = 1.NET
    F(J,N,2) = 0.5*(FA(J,1)+FA(J,2))
   U(J,N,2) = 0.5*(UA(J,1)+UA(J,2))
   V(J,N,2) = 0.5*(VA(J,1)+VA(J,2))
    R(J, N, 2) = 0.5*(RA(J, 1)+RA(J, 2))
    F(J,M.2) = F(J,N.2)
   U(J,M,2) = U(J,N,2)
    V(J,M,2) = V(J,N,2)
    R(J,M,2) = P(J,N,2)
    FC(J) = F(J+N+2)
    UC(J) = U(J,N,2)
    V((J) = V(J,N,2)
    RC(J) = R(J,N,2)
140 CONTINUE
    DETURN
    FND
```

```
SUPPROUTING SOLV3(IT)
   COMMON/RLCO/ NXT.NZT.NX.NZ.NP.NTR.ITMAX.IBDY.NPT.IZIG.ITUPB.ET AF.
                 VGP .A (101) .ETA (101) .DETA (101)
   COMMON/RLC1/ X(101),Z(81),UC(81),FZ(81),P1(81),P2(81),P3(101,81).
                 UF(101,81)
  ,
   COMMON/BLCC/ $1(101),$2(101),$3(101),$4(101),$5(101),$6(101),
                 R1(101).R2(101).R3(101)
  1
   CT4MTN/PLCP/ DELV(101).F(101,81.2).U(101,81.2).V(101,81,2).
                 P(101,81,2)
  1
   COMMON/PRNT/ IPRNT
   TIMENSION A11(101), A12(101), A13(101), A21(101), A22(101), A23(101),
              611(101), G12(101), G13(101), G21(101), G22(101), G23(101),
              W1(101), W2(101), W3(101), DELF(101), DELU(101)
   RFLAX = 1.0
   IF ( IT .GT. 4 ) RELAX = C.50
  W1(1) = 01(1)
  W2(1) = R2(1)
  W3(1) = P3(1)
   \Delta 11(1) = 1.0
   A12(1) = 0.0
   \Delta 13(1) = 0.0
   421(1)= 0.0
   422(1) = 1.0
   123(1)= 0.0
   G11(2) = -1.0
   G12(2)=-0.5*PFT6(1)
   G13(2) = 0.0
   G21(2) = 54(2)
   G23(2)=-2.0*52(2)/0574(1)
   622(2) = 622(2) + 56(2)
   DO 50 7=5*NB
   IF(J .FO. 2) GO TO 10
   DEN
         = (A1?(J-1)*A21(J-1)-A23(J-1)*A11(J-1)-A(J)*
            (612(J-1)*621(J-1)-622(J-1)*611(J-1)))
  1
   G11(J) = \{ (23(J-1)+A(J)+(A(J)+A21(J-1)-A22(J-1)) \} \}
   G12(J)=-(1.0+G11(J)*A11(J-1))/A21(J-1)
   G13(J) = (G11(J)*A13(J-1)+G12(J)*A22(J-1))/A(J)
   C21(J) = (S2(J)*A21(J-1)-S4(J)*A23(J-1)+A(J)*(S4(J)*
            4 22(J-11-56(J)*A 21(J-1))) /DEN
   G22(J) = (4(J)-621(J)*411(J-1))/421(J-1)
   G23(J) = (G21(J) * \Delta 12(J-1) + G22(J) * \Delta 22(J-1) - S6(J))
10 \text{ All(J)} = 1.0
   \Delta 12(J) = -\Delta(J) - G13(J)
   413(J) = 4(J)*G13(J)
   421(J)= 53(J)
   \Delta 22(J) = 55(J) - G23(J)
   423(J) = S1(J) + A(J) + G23(J)
   W1(J) = P1(J)-G11(J)*W1(J-1)-G12(J)*W2(J-1)-G13(J)*W3(J-1)
   W2(J) = R2(J)-G21(J)*W1(J-1)-G22(J)*W2(J+1)-G23(J)*W3(J-1)
   W3(J) = R3(J)
20 COMTINUE
   DELII(NP) = W3(NP)
   F1
             = W1(NP)-412(NP)*DELU(NP)
   = 2
             = W2(NP)-422(MP)*DELU(NP)
   PFLV(NP) = (F2*411(NP)-F1*421(NP))/(A23(NP)*A11(NP)-A13(NP)*
```

421(NP))

```
D^{c}LF(NP) = (F1-A13(NP)*DELV(NP))/A11(NP)
         = NP
30 J
         = J-1
   53
         = W3(J)-PFLU(J+1)+A(J+1)+DELV(J+1)
   PFLV(J) = (A11(J)*(W2(J)+E3*A22(J))-A21(J)*W1(J)+E3*A21(J)*A12(J)
               1/(421(J)*412(J)*4(J+1)-421(J)*413(J)-4(J+1)*
               \Delta 22(J)*\Delta 11(J)*\Delta 23(J)*\Delta 11(J)
   DFLU(J)
             =- 4 (J+1) * DFL V(J) -F3
   DCLF(J) = (W1(J)-^12(J)*DELU(J)-A13(J)*DFLV(J))/A11(J)
   IF(J .GT. 1) GO TO 30
   IF ( IPRMT .LT. 2 ) WRITE(6, 50) V(1,NZ,2), DELV(1)
   00 40 J=1.NP
   F(J,NZ,2)= F(J,NZ,2)+DFLF(J)*RELAX
   U(J\cdot NZ\cdot 2) = U(J\cdot NZ\cdot 2) + DELU(J) + RELAX
40 V(J,NZ,2) = V(J,NZ,2)+DFLV(J)*RELAX
   11(1,NZ,2) = 0.0
   PETURN
```

50 FORMAT(1H , 5X.8HV(WALL) = .F14.6,5X.6HDELVW=.F14.6)
5ND

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16. Abstract	_			
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with Keller's box method, in flows with flow reversal. boundary layers with flow relation of Cebeci and Smith, proposition that unsteady to singularities. We also perbulence model of Bradshaw, equations for both models to predictions with each other. The study reveals that, as layers are free from singularities that the predictions with each other. The study reveals that the predictions of the boundary also reveals that the prediction of the prediction of the boundary also reveals that the prediction of the boundary also reveals the boundary also reveals that the prediction of the boundary also reveals the b	nas been deve In this repo reversal. Us , we consider turbulent boun form turbule Ferriss, and by using the r. in laminar f larities, but layer with i	loped for calc rt, we extend ing the algebra several test ndary layers a nt flow calcul Atwell; we so same numerical lows, the unst there is a cla ncreasing flow th turbulence	ulating unsteathis method to aic eddy visco cases to investo lso remain from the government of the cases of t	ady lamina be turbulent stigate the ee of ng the tur ning compare the t boundary n of rapid ne study
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with Keller's box method, in flows with flow reversal. boundary layers with flow relation of Cebeci and Smith, proposition that unsteady it singularities. We also perbulence model of Bradshaw, equations for both models by predictions with each other. The study reveals that, as layers are free from singularities that the predictions with each other. The study reveals that the predictions of the boundary also reveals that the prediction of the boundary also reveals by Author(s).	nas been deve In this repo reversal. Us , we consider turbulent boun form turbule Ferriss, and by using the r. in laminar f larities, but layer with i	loped for calc rt, we extend ing the algebr several test ndary layers a nt flow calcul Atwell; we so same numerical lows, the unst there is a cl ncreasing flow th turbulence Unlimited	ulating unsteathis method to aic eddy visco cases to investo lso remain from the government of the cases of t	ady lamina be turbulent stigate the ee of ng the tur ning compare the t boundary n of rapid ne study
with Keller's box method, in flows with flow reversal. boundary layers with flow relation of Cebeci and Smith, proposition that unsteady is singularities. We also perbulence model of Bradshaw, equations for both models by predictions with each other. The study reveals that, as layers are free from singularities with each other also reveals that the predictions with each other also reveals that the prediction of the boundary	nas been deve In this repo reversal. Us , we consider turbulent boun form turbule Ferriss, and by using the r. in laminar f larities, but layer with i	loped for calc rt, we extend ing the algebr several test ndary layers a nt flow calcul Atwell; we so same numerical lows, the unst there is a cli ncreasing flow th turbulence Unlimited STAR Cate	ulating unstead this method to aic eddy visco cases to invest lso remain from the government of the car indication reversal. The models are the car indication reversal.	ady lamina b turbulen bsity form stigate the ee of ng the tur- ning ompare the t boundary n of rapid ne study

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